



Solutions to Mat-171 Fall 2020 Recitation

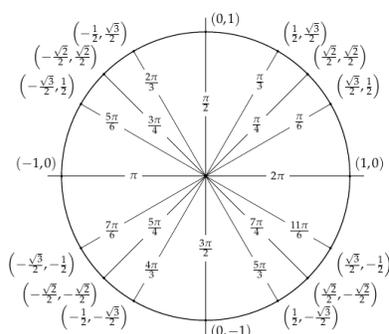


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The slope equals the rise over the run



1

Introduction

This is a document which will go through the answers to the recitation documents. In case you don't like the format of the slides. The first chapter is a review of plotting functions, inverse functions, and algebra.

1.1 Review

1 (a) Factor

$$10x^2 + 11x - 6$$

Answer: Hint: See how we can get 11 from $-6 \cdot 10$.

$$10x^2 + 11x - 6 \rightarrow (10x^2 + 15x) + (-4x - 6)$$

$$\xrightarrow{\text{get terms in paranthesis same}} 5x(2x + 3) - 2(2x + 3)$$

$$(5x - 2) \cdot (2x + 3)$$

$$x = \frac{2}{5}, x = -\frac{3}{2}$$

(b) Factor

$$6x^3 + 9x^2 - 60x$$

Answer:

$$\begin{aligned}
 6x^3 + 9x^2 - 60x &= 3x(2x^2 + 3x - 20) \\
 &= 3x \left(2x^2 \underbrace{-5x + 8x}_{=3x} - 20 \right) \\
 &= 3x(x(2x - 5) + 4(2x - 5)) \\
 &= 3x((x + 4) \cdot (2x - 5))
 \end{aligned}$$

So, our solutions are $x = 0$, $x = \frac{5}{2}$, and $x = -4$.

(c) Factor

$$x^3 + 4x^2 - 9x - 36$$

Answer:

$$\begin{aligned}
 x^3 + 4x^2 - 9x - 36 &= x^2(x + 4) - 9(x + 4) \\
 &= (x^2 + 9) \cdot (x + 4) \\
 &= (x + 3) \cdot (x - 3) \cdot (x + 4)
 \end{aligned}$$

So, our solutions are $x = 3$, $x = -3$, and $x = -4$. Notice, here we factored the terms separately. This process is all about writing the polynomial in an "easier" form.

2 (a) Simplify

$$\frac{x^2 - 2x - 8}{x^2 - 9} \div \frac{x - 4}{x + 3}$$

Answer: Notice, $x^2 - 9 = (x + 3) \cdot (x - 3)$ and $x^2 - 2x - 8 = (x - 4) \cdot (x + 2)$, so we have

$$\frac{(x - 4) \cdot (x + 2) x + 3}{(x + 3) \cdot (x - 3) x - 4} = \frac{x + 2}{x - 3}$$

(b) Simplify

$$\frac{\frac{1}{x+h} - \frac{1}{x}}{h}$$

Answer: The trick here is to match the denominators:

$$\begin{aligned}
 &\frac{1}{h} \cdot \left(\frac{x}{x(x+h)} - \frac{x+h}{x(x+h)} \right) \\
 &= \frac{1}{h} \cdot \left(\frac{-h}{x(x+h)} \right) \\
 &= \boxed{\frac{-1}{x(x+h)}}
 \end{aligned}$$

(c) Simplify

$$\frac{\frac{3}{x-2} - \frac{4}{x+2}}{\frac{7}{x^2-4}}$$

Answer: Split the numerator and denominator to make things clearer, noting dividing in denom is same as multiplying in numerator and $(x-2) \cdot (x+2) = x^2 - 4$:

$$\begin{aligned} & \frac{3(x+2) - 4(x-2)}{x^2-4} \cdot \frac{x^2-4}{7} \\ &= \frac{-x+14}{7} \end{aligned}$$

(d) Simplify

$$\frac{x-5}{\sqrt{x+11}-4}$$

Answer: The trick here is to rationalize the denominator, i.e. multiply by the denominator but flip the - to a +, so as to cancel terms when squaring!

$$\begin{aligned} & \frac{x-5}{\sqrt{x+11}-4} \cdot \left(\frac{\sqrt{x+11}+4}{\sqrt{x+11}+4} \right) \\ &= \frac{(x-5) \cdot (\sqrt{x+11}+4)}{x+11-16} \\ &= \sqrt{x+11}+4 \end{aligned}$$

3 Is the following equality true?

$$\sqrt{x^2+y^2} = x+y$$

Answer: No. For example, let $x=2, y=3$. Then $\sqrt{x^2+y^2} = \sqrt{13}$, but $x+y=5$, so they are not equal. Additionally, note after squaring both sides, we have

$$(x+y)^2 = x^2 + y^2 + 2xy \neq x^2 + y^2$$

so the two are not equal.

1.2 Week 2

- Graph the function $f(x) = \sqrt{x}$. Then graph the function $g(x) = \frac{1}{2}\sqrt{x+2} - 2$ using transformations. List the transformations and give the domain and range of $g(x)$. Make sure you label your graph.**

Answer: See the figure (1.1) below: The domain for $f(x)$ is $(0, \infty)$, the range is $[0, \infty)$, the domain of $g(x)$ is $(-2, \infty)$ and the range is $[-2, \infty)$.

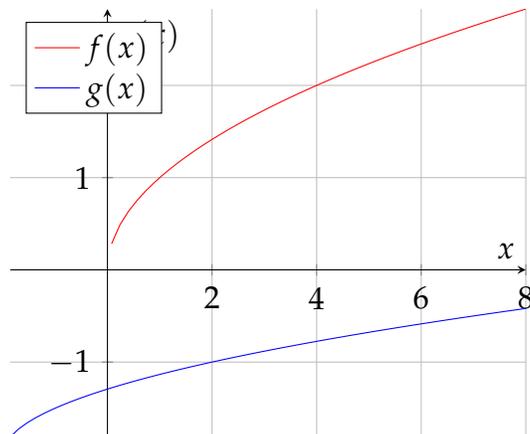


Figure 1.1: Week 2, question 1.

2. Find the function $g(r) = \frac{4r^2-1}{r^2}$ when $r = -x$. Explain why this shows that $g(r)$ is an even function.

Answer: Here we plug in $-x$ where ever we see r . The tricky part is to keep an eye on the negative sign. For this reason, its suggested to write $r = (-x)$. Lets plug in and see what we get:

$$\begin{aligned} g(-x) &= \frac{4(-x)^2 - 1}{(-x)^2} \\ &= \frac{4x^2 - 1}{x^2} \end{aligned}$$

This is an even function because $h(x) = h(-r)$ (just calling x r here.)

3. Find and simplify the difference quotient of the function $f(x) = -x^2 + 2x - 1$.

Answer: The difference quotient is

$$\frac{f(x+h) - f(x)}{h}$$

We know $f(x)$, but we need to calculate $f(x+h)$. Let's do it.

$$\begin{aligned} f(x+h) &= -(x+h)^2 + 2(x+h) - 1 \\ &= -x^2 - h^2 - 2xh + 2x + 2h - 1 \end{aligned}$$

Then, we have that the difference quotient is

$$\begin{aligned} \frac{f(x+h) - f(x)}{h} &= \frac{-x^2 - h^2 - 2xh + 2x + 2h - 1 - (-x^2 + 2x - 1)}{h} \\ &\xrightarrow{\text{distribute the negative sign, and combine terms of same order}} \\ &= \frac{-x^2 + x^2 - h^2 + -2hx + 2h + 2x - 2x - 1 + 1}{h} \\ &= \frac{-h^2 - 2hx + 2h}{h} \\ &= \boxed{-h - 2x + 2} \end{aligned}$$

4. The function $h(t) = -4t^2 + 8t + 32$ models the height, h , of a thrown ball after t seconds.

- (a) Find the y -intercept and interpret it in the context of the problem.

Answer: We find the y -intercept by setting time, t , equal to 0. Therefore the intercept is 32, which we interpret as the height you throw the ball from.

- (b) Find the interval of time where the ball's height is decreasing.

Answer:

There are multiple ways to do this. For one, we can plot and see where the maximum of the function is and find the time where this occurs. See figure(1.2). Using calculus, you can find the derivative of $h(t)$ with respect to time, and then see when that is negative.

$$h'(t) = \frac{dh}{dt} = -8t + 8$$

which is negative when $t > 1$. Therefore, after 1 second the height decreases.

However, looking at the plot, 1.2, is probably more in line with what we've just learned. There, we can see the interval the ball is decreasing is t in $[1, 4]$.

Also, you can find the vertex from $-b/2a$, to get x -coordinate, plug back in at x to get y .

- (c) What is the domain and range of this function in the context of the problem?

Answer: The domain we can think of as the t values, the time the ball is thrown. We start at 0. Similarly, we do not have negative distance, the ground is 0. Therefore, the maximum number of time is the time at which the ball reaches the ground which is at $t = 4$. We can find this by factoring the equation:

$$\begin{aligned} -4t^2 + 8t + 32 &= -4(t^2 - 2t - 8) = 0 \\ &= -4((t^2 - 4t) + (2t - 8)) = 0 \\ &= -4(t + 2)(t - 4) = 0 \end{aligned}$$

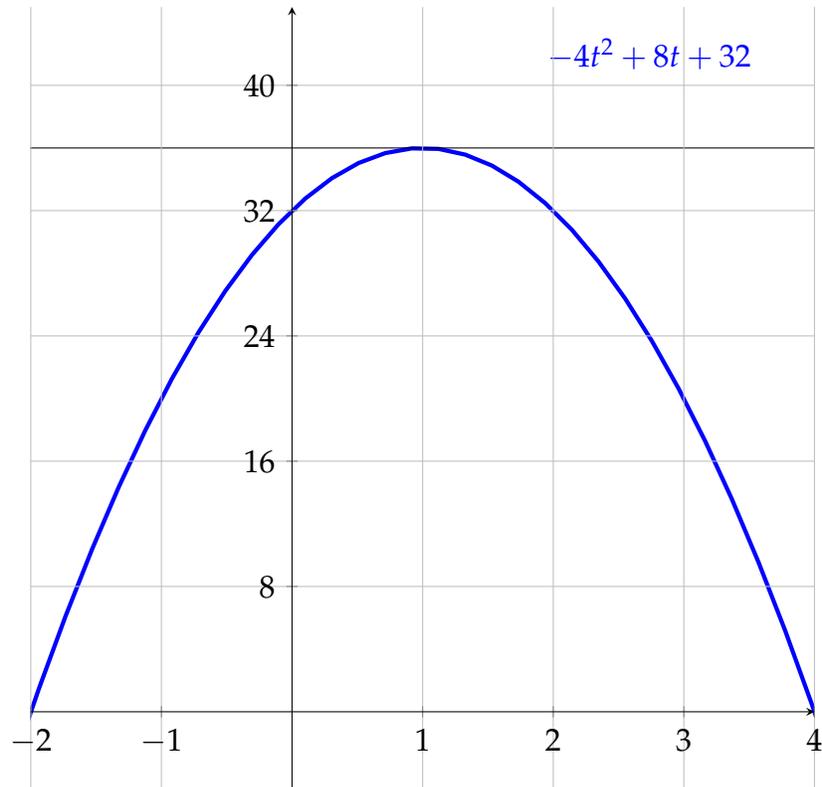


Figure 1.2: Week 2, question 4(a)

So our two solutions are at $t = -2$ and $t = 4$, but our domain tells us $t \geq 0$. So the maximum time is $t = 4$, meaning our domain is $t \in [0, 4]$. The range has a minimum height of 0. The maximum height occurs at $t = 1$, as seen from (a). Then plugging in $h(1) = -4 + 8 + 32 = 36$, so our maximum height is 36. Plotting this is also a good idea. Again, reference figure(1.2).

1.3 Week 3

1. In the theory of relativity, the mass of a particle with speed v is

$$m = f(v) = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Where m_0 is the rest mass of the particle and c is the speed of light in a vacuum. Note that m_0 and c are constants.

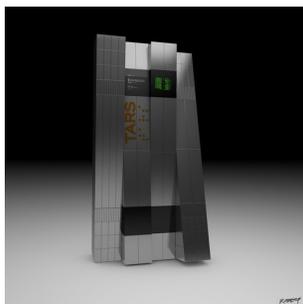


Figure 1.3: TARS from interstellar... that whole movie was essentially based on this equation, but in a more general and complicated form. Click on picture for link to cool youtube video explaining relativity (the special case at least).

- (a) What is the domain of the function $f(v)$?

Answer: The minimum speed we can have is $v = -c$. The maximum is $v = c$. So the domain is $(-c, c)$, note the open brackets. What does this mean? You cannot under any circumstance travel faster than the speed of light.

- (b) Find the inverse of $f(v)$.

Answer: Remember, if $f^{-1}(m) = v$, then $f(v) = m$. Let's do it:

$$\begin{aligned} m &= \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}} \\ \left(\frac{m}{m_0}\right)^2 &= \frac{1}{1 - \frac{v^2}{c^2}} \quad \xrightarrow{\text{note}} \quad 1 - \frac{v^2}{c^2} = \frac{c^2}{c^2 - v^2} \\ \left(\frac{m}{m_0}\right)^2 &= \frac{1}{\frac{c^2 - v^2}{c^2}} \\ &= c^2 - c^2 \left(\frac{m_0^2}{m^2}\right) = v^2 \\ v &= c \sqrt{1 - \frac{m_0^2}{m^2}} \end{aligned}$$

so the inverse function is:

$$v = f^{-1}(m) = c\sqrt{1 - \left(\frac{m_0}{m}\right)^2}$$

(c) **What is the domain of $f^{-1}(m)$?**

Answer: The domain is the limits of m . The minimum m is m_0 , since beneath that would mean we would be taking the square root of a negative number, which does not make sense here. The maximum m is ∞ , in that case, $v \rightarrow c$.

So the domain is $[m_0, \infty)$.

2. A perfectly spherical balloon is being filled with air. The radius increases at a rate of 3 inches/sec. We can express the radius of the balloon as the function $R(t) = 3t$, where t is represents time in seconds. We can also express the volume of the balloon as the function $V(r) = \frac{4}{3}\pi r^3$, where r represents the radius in inches.

(a) **Give a function in terms of t for the volume of the balloon (find $V(r, t)$).**

Answer: We know $V(r) = 4/3\pi r^3$. However, here our radius changes over time, so maybe it makes more sense to write the volume as a function of the radius and the time, that is

$$V(r, t) = 4/3\pi(R(t))^3 = 4/3\pi(3t)^3$$

So there we have it.

(b) **What is the volume at $t = 3$?**

Answer: Plug in $t = 3$. Doing so yields:

$$V(r, 3) = \frac{4}{3}\pi(3 \cdot 3)^3 = \frac{4}{3}\pi \cdot 729 = 972\pi \text{ inches}^3$$

(c) **Find $V^{-1}(t)$ and explain its meaning (where $V(t)$ is the function describing the volume).**

Answer: We do what we did in 1(b), where we swap out t and $V(t)$, which we'll just call V for easiness.

$$\begin{aligned} V &= 4/3\pi 27t^3 = 36\pi t^3 \\ \frac{V}{36\pi} &= t^3 \\ t &= \left(\frac{V}{36\pi}\right)^{1/3} \\ \text{therefore} \rightarrow V^{-1}(t) &= \left(\frac{t}{36\pi}\right)^{1/3} \end{aligned}$$

What does this mean? If we know our volume, we can find out how long the balloon has been being filled. Cool!

3. Simplify:

(a) $\frac{x^2+19}{2-x}$ where $x = 3i$.

Answer: To solve this, remember

$$i = \sqrt{-1} \quad (1.1)$$

$$i^2 = -1 \quad (1.2)$$

$$i^3 = -i \quad (1.3)$$

$$i^4 = 1 \quad (1.4)$$

Pretty cool. Then,

$$\begin{aligned} \frac{x^2 + 19}{2 - x} &= \frac{(3i)^2 + 19}{2 - x} \\ &= \frac{-9 + 19}{2 - x} \\ &= \boxed{\frac{10}{2 - 3i}} \end{aligned}$$

we can simplify further to get i in the numerator. Multiply by $\frac{2+3i}{2+3i}$, the conjugate, which is really just a fancy way of multiplying by 1 to get rid of some not so nice terms, then

$$\begin{aligned} \frac{10}{2 - 3i} \cdot \frac{2 + 3i}{2 + 3i} &= \frac{20 + 30i}{4 - 6i + 6i + 9} \\ &= \boxed{\frac{20 + 30i}{13}} \end{aligned}$$

(b) $\frac{x^2+11}{3-x}$ where $x = 4i$.

Answer:

$$\begin{aligned} \frac{x^2 + 11}{3 - x} &= \frac{(4i)^2 + 11}{3 - x} \\ &= \frac{-16 + 11}{3 - x} \\ &= \boxed{\frac{-5}{3 - 4i}} \end{aligned}$$

Again, multiply by the conjugate

$$\begin{aligned} \frac{-5}{3 - 4i} \cdot \frac{3 + 4i}{3 + 4i} &= \frac{-15 - 20i}{9 + 16} \\ &= \boxed{\frac{-15 - 20i}{25}} \end{aligned}$$

Despite the name, imaginary numbers are very much “real” and quite useful.



2

Complex Numbers and Stuff

2.1 Week 4

1. Use long division to divide $6x^3 + 11x^2 - 31x + 15$ by $3x - 2$.

Answer:

$$\begin{array}{r}
 2x^2 + 5x - 7 \dots \\
 \hline
 3x - 2 \left) 6x^3 + 11x^2 - 31x + 15 \\
 \underline{6x^3 - 4x^2} \\
 15x^2 - 31x + 15 \\
 \underline{15x^2 - 10x} \\
 -21x + 15 \\
 \underline{-21x + 14} \\
 1
 \end{array}$$

Therefore the answer is $2x^2 + 5x - 7 + \frac{1}{3x-2}$.

- (a) Use synthetic division to check your answer:

Answer: Since we are dividing by a linear factor, this is totally cool. We

multiply by $1/3$ to get rid of the 3, then $2/3$ is the remainder:

$$\begin{array}{r|rrrr} & x^3 & x^2 & x & 0 \\ \frac{2}{3} & 2 & \frac{11}{3} & -\frac{31}{3} & 5 \end{array}$$

Then, we note

$$\begin{array}{r|rrrr} \frac{2}{3} & 2 & \frac{11}{3} & -\frac{31}{3} & 5 \\ & & \frac{4}{3} & \frac{10}{3} & -\frac{14}{3} \\ \hline & 2 & 5 & -7 & \frac{1}{3} \end{array}$$

therefore, the quotient is $2x^2 + 5x - 7$ with remainder $\frac{1}{3}$. Therefore, we have

$$2x^2 + 5x - 7 + \frac{1/3}{x - 2/3}$$

where we did the first step by multiplying $\frac{2}{3}$ by 2, then we add this total to $\frac{11}{3}$. Then this sums to 5, which we multiply by $\frac{2}{3}$ and then sum to $-\frac{31}{3}$, the next term in the top row. And so on. Note, we sum here not subtract like in the first question.

2. Explain the relationship between the multiplicity of a zero and whether or not the graph crosses or touches the x -axis at that zero.

Answer: The multiplicity refers to the number of times the factor appears in the polynomial, for example imagine we have a factor $(x + 1)^3$ means -1 has the multiplicity 3. If the multiplicity is even, we do not cross the x -axis but just touch it, whereas if its odd, we go through. See the figure 2.1

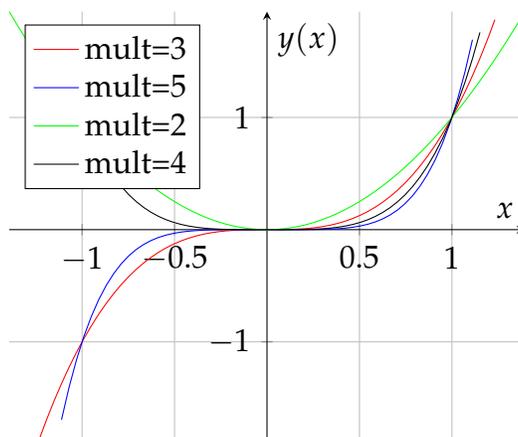


Figure 2.1: Week 4, question 2.

3. One root of a certain polynomial is $2 + 3i$. Another root must exist for this polynomial. What is it?

Answer: Imaginary zeros *always* come in complex conjugate pairs, which means $2 - 3i$ is also a zero.

4. Describe a strategy for graphing a polynomial function. In your description, mention intercepts, multiplicity, the polynomial's degree, and the leading coefficient.

Answer: If quadratic, find the vertex from $-b/2a$, to get x -coordinate, plug back in at x to get y . The intercepts, x and the y , give us a place to start, the degree dictates the width. Factor out anything that seems like an obvious factor. For example, let $g(x) = 4x^4 + x^3 - 15x^2 - 3x + 0$. We see that $x = -1$ is a root. Divide out $x + 1$, long or synthetic division, and get $g(x) = (x + 1)(4x^3 - 3x^2 - 12x + 9)$. We factor the inner term:

$$\begin{aligned} 4x^3 - 3x^2 - 12x + 9 &= x^2(4x - 3) - 3(4x - 3) \\ &= (4x - 3)(x^2 - 3) \\ &= (4x - 3)(x + \sqrt{3})(x - \sqrt{3}) \end{aligned}$$

The plot is in figure 2.2 The degree tells you how many local max/min you have, i.e.

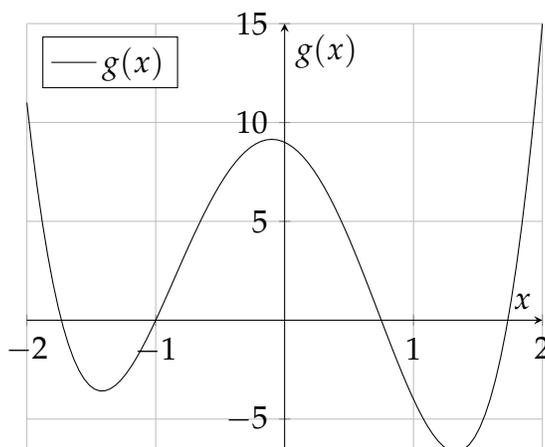


Figure 2.2: Week 4, question 4.

degree with polynomial n has $n - 1$ turning points. Further, the leading coefficient in conjunction with the degree dictates the end behavior of the graph:

Case	End Behavior
n is odd, a_n is positive	Graph falls on the left and rises on the right
n is odd, a_n is negative	Graph rises on the left and falls on the right
n is even, a_n is positive	Graph rises on the left and rises on the right
n is even, a_n is negative	Graph falls on the left and falls on the right

Table 2.1: Week 4. Question 4. Leading coefficient test. Let n be the degree and a_n be the leading coefficient.

5. Use what was discussed in question 4 to plot the polynomial $f(x) = -2(x + 3)^2(x - 5)$.

Answer: See figure 2.3.

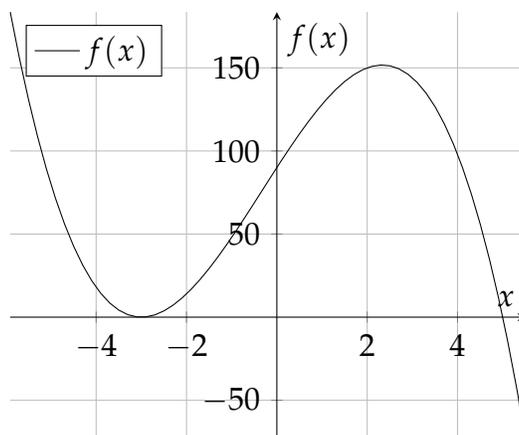


Figure 2.3: Week 4, question 5.

6. Consider a 3rd degree polynomial. Construct the polynomial if 6 and $-5 + 2i$ are zeros and $f(2) = -636$.

Answer: We know that if $-5 + 2i$ is a root, then so is $-5 - 2i$. So we can write the zeros as linear factors, multiplying by a to ensure that the $f(2)$ condition is satisfied:

$$\begin{aligned}
 f(x) &= a(x - 6)(x - (-5 + 2i))(x - (-5 - 2i)) \\
 &= a(x - 6)(x + 5 - 2i)(x + 5 + 2i) \\
 \xrightarrow{\text{Term by term for last 2}} &= a(x - 6)(x^2 + 5x + 2ix + 5x + 25 + 10i - 2ix - 10i - 4i^2) \\
 \xrightarrow{i^2 = -1} &= a(x - 6)(x^2 + 10x + 29) \\
 &= a(x^3 + 10x^2 + 29x - 6x^2 - 60x - 174) \\
 &= a(x^3 + 4x^2 - 31x - 174) \\
 \xrightarrow{\text{plug in } x=2} &= a(8 + 16 - 62 - 174) = -636 \quad \text{from problem} \\
 a \cdot (-212) &= -636 \\
 a &= 3
 \end{aligned}$$

Therefore, now that we have a , we have that our third degree polynomial is:

$$3(x^3 + 4x^2 - 31x - 174) = 3x^3 + 12x^2 - 93x - 522$$

2.2 Week 5

1. Consider the rational function: $f(x) = \frac{x^2 - 2}{2x - 6}$.

(a) What is the vertical asymptote, $x = a$.

Answer: To find the vertical asymptote, let $2x - 6 = 0$ which means the vertical asymptote is at $x = 3$. **Use a calculator to find:**

i. The maximum of $f(x)$ when $x > a$

Answer: The derivative is given by $\frac{x^2-6x+6}{2(x-3)^2}$ set equal to 0.

Do the quadratic formula to find root of numerator: $\frac{6 \pm \sqrt{12}}{2}$ around (.35, .35)

ii. The minimum of $f(x)$ when $x > a$

Answer: around $x = 5.65, y = 5.65$

(b) What is the horizontal asymptote, $y = b$?

Answer: If the degree of the denominator is **bigger** than the degree of the numerator, the horizontal asymptote is the x-axis, i.e. $y = 0$.

If the degree of the denominator is **smaller** than the numerator, there is no horizontal asymptote.

If the degrees of denominator and numerator are the same, the horizontal asymptote equals the leading coefficient (the coefficient of the largest exponent) of the numerator divided by the leading coefficient of the denominator.

So none here.

2. Consider the rational function: $g(x) = \frac{x+4}{x^2+x-12}$

(a) Find the x intercept(s) and the y intercept.

Answer: Factor $x^2 + x - 12$, which yields $(x - 3)(x + 4)$, so

$$g(x) = \frac{x + 4}{x^2 + x - 12} = \frac{1}{x - 3}$$

The y intercept is when $x = 0$ which is $y = -1/3$. Meanwhile the x -intercept does not exist because the x term is in the denominator, so there is not value of x that will allow us to have $g(x) = 0$, well except for $\pm\infty$, but even then there is no crossing of the x -axis.

(b) What is the domain of the function? With this information find any vertical asymptotes. How many are there and why?

Answer: The domain is $(-\infty, -4) \cup (-4, 3) \cup (3, \infty)$. So the vertical asymptote is at $x = 3$. Just like question 1...interesting.

(c) Where is the hole in the graph?

Answer: A **vertical asymptote** occurs when a factor of the denominator of a rational expression does not cancel with a factor from the numerator.

A **hole**, on the other hand, occurs where the factors cancel out. So here that would be at $x = -4$. What is $g(-4)$? We do this with the simplified form, $\frac{1}{x-3} = \frac{1}{-4-3} = -\frac{1}{7}$

(d) Find any horizontal asymptotes.

Answer: $y = 0$ by the reasoning from part 1. We can also take the limit as $x \rightarrow -\infty$ and $x \rightarrow \infty$ and see what that equals, which is $y = 0$.

(e) Describe the behavior of x as $g(x)$ approaches the horizontal asymptote.

Answer: It gets over so close.

(f) Describe the behavior of $g(x)$ as x approaches a vertical asymptote.

Answer: Same thing. Goes down from left, up from right. Basically parallel to the asymptote at infinity.

(g) Graph the function.

Answer:

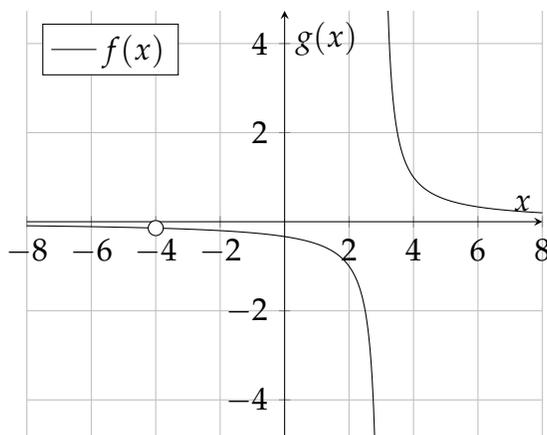
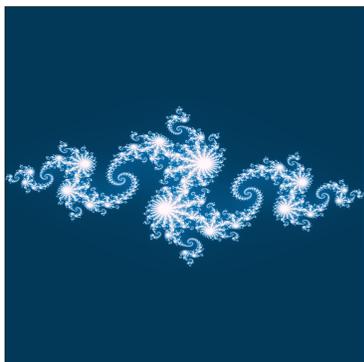


Figure 2.4: Week 5, question 1.

Let $w_1 \in W_1$ and $w_2 \in W_2$. Then they're both in S by the problem...and then because S is a subspace, then $w_1 + w_2 \in S$ it's closed under addition, so it contains $w_1 + w_2$ and since those were arbitrary elements.

Click on [this link](#) for a cool video about the picture below.



3

Exponential Functions...and more

3.1 Week 6

1. The general model for growth of a population is the following:

$$P(t) = P_0 e^{rt} \tag{3.1}$$

Where $P(t)$ is the population after a certain amount of time, P_0 is the initial population, r represents the growth (or decay) rate, and t represents the amount of time.

- (a) A scientist starts with 100 bacteria in an experiment. After 5 days, they discover that the population has grown to 350. Find the growth rate.

Answer: We know that $P(5) = 350$. We also know that $P_0 = 100$. Therefore, we have

$$\begin{aligned} P(t) &= P_0 e^{rt} \\ P(5) &= 350 = 100 \cdot e^{r \cdot 5} \\ \frac{350}{100} &= e^{5r} \end{aligned}$$

take natural log to get rid of exp on right $\rightarrow \ln(3.5) = 5r$

$$r = \frac{\ln(3.5)}{5} \approx 0.25$$

(b) What is the population after 15 days?

Answer: We have all the components of the equation from the answer in part(a) and what was given in the problem. We have (plugging in $t = 15$)

$$P(15) = 100 \cdot e^{0.25 \cdot 15} = 4252$$

Note, we round down to the integer because we can't have "part of a bacteria".

(c) When is the population 1000?

Answer: Here, we are given the left hand side of the equation(3.1), which is 1000. The question we must now solve for is t . We can solve this because we already solved for r in part(a). Using the properties of the logarithm (like in part(a)):

$$\begin{aligned} 1000 &= 100 \cdot e^{0.25 \cdot t} \\ \frac{1000}{100} &= e^{0.25 \cdot t} \\ 10 &= e^{0.25 \cdot t} \\ \xrightarrow{\text{take natural log to get rid of exp on right}} \ln(10) &= 0.25t \\ \frac{\ln(10)}{0.25} &= t \end{aligned}$$

plugging in the value of $\ln(10)/0.25$ yields time 9.21 days. So after 9 days (and about 5 hours), our population is 10 times bigger than the original.

2. Consider the function

$$g(x) = -2e^{2x-3} - 3 \tag{3.2}$$

(a) Describe transformations on $f(x) = e^x$ that will produce the function $g(x)$.

Answer: So we begin with $f(x) = e^x$. The transformation is of the form

$$b \cdot e^{ax+c} + d$$

These transformations do the following:

- Vertically shift the function d units in the same direction as d , creating a new horizontal asymptote. That is, $f(x) - c$ shifts c units down, and $f(x) + c$ shifts c units up.
- Horizontally c units in the opposite direction of c . That is, $f(x - c)$ shifts c units to the right, and $f(x + c)$ shifts c units to the left.
- $-f(x)$ reflects the function over the x-axis, and $f(-x)$ reflects the function over the y-axis.
- $b \cdot f(x)$, where $b > 1$ stretches the curve by b and $0 < b < 1$ shrinks the curve by a factor of b vertically.
- $f(ax)$, where $0 < a < 1$, stretches the curve horizontally by a factor of a .

- $f(ax)$ where $a > 1$, shrinks the curve horizontally by a factor of a .

First, we draw the horizontal asymptote, which occurs at $y = -3$. Then, multiplying by -2 stretches the curve vertically by a factor of two, and the negative sign tells us that we reflect the function over the y -axis. Now, we turn our attention to the arguments within the exponent, i.e. those that directly “interact” with x . Because we multiply x by 2 , we shrink the curve horizontally by a factor of 2 . Finally, we shift the function $3/2$ units to the left, because $c = 3$ and the 2 in front of x (think of the change in x -intercept when $f(x) = x$ and $f(x) = 2x - 3$), and we shift to the opposite direction. Figure 3.1 shows the two graphs.

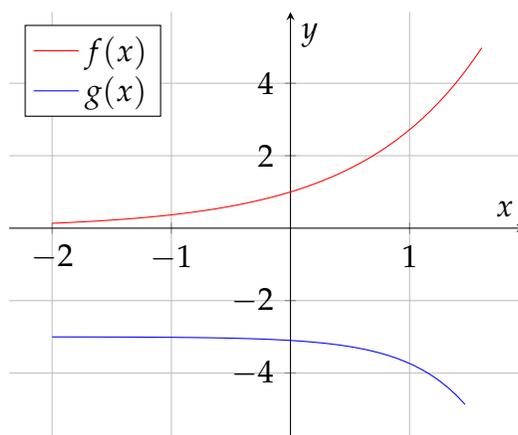


Figure 3.1: Week 6, question 2.

- (b) What is the range, horizontal asymptote, and y intercept of $f(x)$. How do these change when $f(x)$ is transformed to $g(x)$.

Answer: The range is $(0, \infty)$, because the horizontal asymptote is at $y = 0$, and by the shape we see it always increases (see figure 3.1), horizontal asymptote at $y = 0$, and no x -intercept, y -intercept is at $(0, 1)$. However, when we transform, we see that the horizontal is at $y = -3$ (because of the vertical shift down). The y -intercept is calculated when $x = 0$, which is

$$y = g(0) = -2e^{0-3} - 3 \approx -3.1$$

Finally, the range is now from $(-\infty, -3)$, because of the horizontal asymptote.

- (c) Find the inverse function $g^{-1}(x)$ of $g(x)$.

Answer: Recall, that the inverse function can be solved as follows:

$$g^{-1}(x) = y \leftrightarrow g(y) = x$$

Inverses are annoying :(but here goes. First, recall

$$g(x) = -2e^{2x-3} - 3$$

Now, let $g(x) = y$, because then it'll be a little easier to see what is going on. We must now solve for x in terms of y .

$$\begin{aligned}
 y &= -2e^{2x-3} - 3 \\
 y + 3 &= -2e^{2x-3} \\
 -\frac{y+3}{2} &= e^{2x-3} \\
 \ln\left(-\frac{y+3}{2}\right) &= 2x - 3 \\
 \xrightarrow{\text{properties of logs}} \ln(-(y-3)) - \ln(2) &= 2x - 3 \\
 \frac{\ln(-(y-3)) - \ln(2) + 3}{2} &= x
 \end{aligned}$$

Since the function $g(x)$ maps x to y , then the inverse function $g^{-1}(x)$ maps y back to x . So, we need to *take* x as an argument. Therefore, we have to interchange the x and y variables. Expanding the logarithm out was just to reiterate the properties of the logarithm (more specifically the natural logarithm here). The answer is probably cleaner as

$$g^{-1}(x) = \frac{\ln\left(-\frac{x+3}{2}\right) + 3}{2}$$

(d) What is the domain of $g^{-1}(x)$?

Answer: The domain is $(-\infty, -3)$. This is because we cannot have the natural log of 0 or any negative number. Additionally, the domain is the range of $g(x)$, which we solved in 2(b).

(e) We can see $g^{-1}(x)$ is a transformation of $h(x) = \ln(x)$. What is the domain of $h(x)$ and how does the domain change with a horizontal shift to $h(x)$?

Answer: The domain of $\ln(x)$ is $(0, \infty)$. If we shift $\ln(x+c)$, we shift c units to the left if $c < 0$ and c units to the right if $c > 0$.

3.2 Week 7

1. The following functions model interest in two ways:

$$\text{Compounding Periods} \longrightarrow A(t) = P\left(1 + \frac{r}{n}\right)^{nt}$$

$$\text{Continuous compounding} \longrightarrow A(t) = Pe^{rt}$$

Where

- $A(t)$ represents the amount of money after a certain amount of time, t .
 - P is the amount of money you start with.
 - r represents the interest rate.
 - n is the number of times interest is compounded in one year.
 - t represents the amount of time in years.
- (a) A child is born and their parents put \$1000 in a savings account that compounds every month ($n=12$ times a year). On the child's 18th birthday, they receive 4200.57 after closing the account.
- i. At what interest rate was this account collecting compound interest? Use the compounding period equation.

Answer: What do we know. Well, we start at $P = 1000$. Also, we know that $n = 12$. Similarly, we know that $t = 18$ years. And, we know that $A(t) = 4200.57$. So, we have everything we need but r and will solve for that. First, we do this analytically and then plug in the numbers.

$$A(t) = P \left(1 + \frac{r}{n} \right)^{nt}$$

$$\frac{A(t)}{P} = \left(1 + \frac{r}{n} \right)^{nt}$$

$$\xrightarrow{\text{take log}} \ln \left(\frac{A(t)}{P} \right) = nt \ln \left(1 + \frac{r}{n} \right)$$

$$\frac{\ln \left(\frac{A(t)}{P} \right)}{nt} = \ln \left(1 + \frac{r}{n} \right)$$

$$\xrightarrow{\text{re-exponentiate}} e^{\ln \left(\frac{A(t)}{P} \right) / nt} = 1 + \frac{r}{n}$$

$$n \left(e^{\ln \left(\frac{A(t)}{P} \right) / nt} - 1 \right) = r$$

$$n \left(\frac{A(t)}{P} e^{\frac{1}{nt}} - 1 \right) = r$$

The last two lines are equivalent, and it is really a preference which form you'd like it in. The last line is really to emphasize properties of exponentials, that $e^{ab} = e^a e^b = e^{a+b}$, and that $e^{\ln(a)} = a$. Now, plugging in the numbers, we have

$$r = \frac{e^{1.43/(12*18)} - 1}{12} = .08 \rightarrow 8\% \text{ interest rate}$$

So the interest rate is 8.00%.

- ii. If this account was continuously compounded at the same rate, how much money would be in the account after $t = 18$ years.

Answer: We use $r = .08$ from part a. We plug in to the continuous compounding equation and find:

$$A(18) = 1000 \cdot e^{.08 \cdot 18} = 4220.69$$

- (b) You decide to invest \$8000 for 6 years and you have a choice between two accounts. The first pays 7% per year, compounded monthly. The second pays 6.85% per year, compounded continuously. Which is the better investment.

Answer: We plug in the two equations, keeping in mind how to convert between percentages and decimals:

$$\text{Compounding Periods} \rightarrow A(t) = P \left(1 + \frac{r}{n} \right)^{nt} = 8000 \left(1 + \frac{0.07}{12} \right)^{12 \cdot 6} = \$12,160.84$$

$$\text{Continuous compounding} \rightarrow A(t) = P e^{rt} = 8000 \cdot e^{.0685 \cdot 6} = \$12,066.60$$

Narrowly, the compounded interest is better.

2. Plot $y = 2^x$ and $x = 2^y$ on the same axis. What is their relationship? Hint:

$$x = 2^y \implies y = \log_2(x)$$

Answer: On your calculator, to change the base go to math and scroll to logbase. If this isn't an option, remember that

$$\log_b(x) = \log(x) / \log(b) = \ln(x) / \ln(b)$$

We can use \ln instead of \log because its just the ratio we care about. In our case, plot $x = 2^y \implies y = \ln(x) / \ln(2)$ and it's reflected over the $y = x$ line

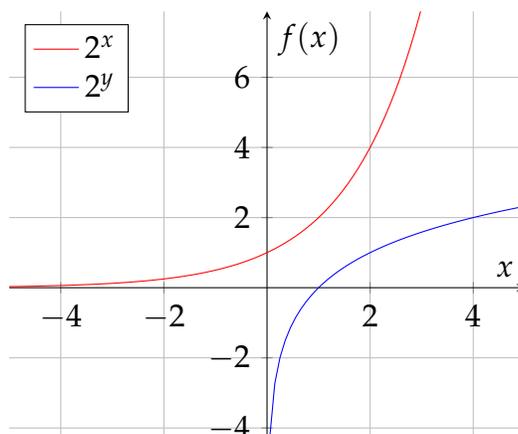


Figure 3.2: Week 7, question 2.

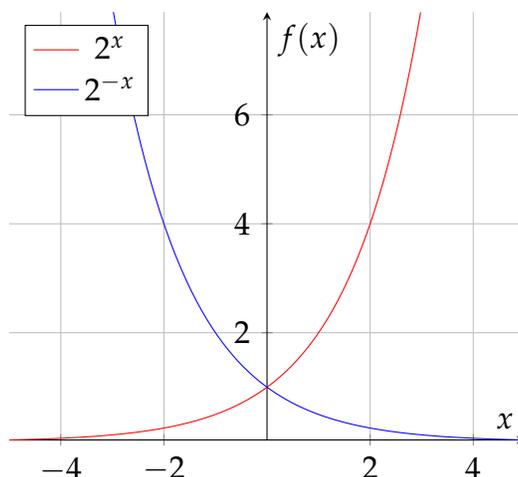


Figure 3.3: Week 7, question 3.

3. Plot $y = 2^x$ and $y = 2^{-x}$ together and describe how they are related.

Answer: This one is a little more straight forward. Reflection over the y-axis.

4. Radioactive materials decay according to

$$f(t) = e^{kt}$$

where t is the time, $f(t)$ is how much of the radioactive material is left, and k is the decay constant. The constant out front here is 1, because the initial amount of the substance is the entire substance, and we want to calculate what percentage of it is left after a time t . A half-life is the amount of time it takes for half the material to radioactively decay, i.e. the time t , where $f(t) = 0.5$. Bananas are a good source of potassium, which is a healthy food. But potassium is also radioactive with a half life of 1 billion years. Find the decay constant of a banana.

Answer: We plug in 0.5 (or $1/2$) for $f(t)$.

$$0.5 = e^{k \cdot 1 \times 10^9} \longrightarrow \frac{\ln(0.5)}{1 \times 10^9} = k$$

which means

$$k = -6.93 \times 10^{-10}$$

Which makes sense that its negative in this context.

5. What is the domain of $f(x) = \log_3(x^2 - 5x + 6)$, where the 3 subscript denotes that we are in base 3 for our logarithm.

Answer: The base 3 here is not very important. That matters were we to exponentiate, say if

$$\log_3(x) = 2 \implies x = 3^2$$

We first have to factor $x^2 - 5x + 6$. Why do we factor? Well if we have $(x - a)(x - b)$, we can see where $x = a$ and $x = b$ and that will yield a negative number. The log

of a negative number is not defined so the function does not exist in that domain interval. Notice, that because $-5 = -3 + -2$, and $-3 \cdot -2 = 6$, then

$$x^2 - 5x + 6 = 0 \longrightarrow (x - 3)(x - 2)$$

Then we have

$$\log_3((x - 3)(x - 2))$$

Therefore, we have roots at $x = 2$ and $x = 3$, which is an issue when taking the logarithm. We therefore have interval divisions, and have to test each interval. First, we check $x < 2$. At $x = 0$, we have $(0)^2 - 5 \cdot 0 + 6 > 0$, so we satisfy the inequality.

Now for the interval $2 < x < 3$, we choose $x = 2.5$. Then

$$f(2.5) = 2.5^2 - 5 \cdot 2.5 + 6 = -0.25 < 0$$

therefore, since we *cannot* pass a negative number as an argument to a log argument. So between $x = 2$ and $x = 3$ there is no output.

Finally, if $x > 3$, we want to show that $f(x) > 0$. Choose $x = 4$.

$$f(4) = 4^2 - 5 \cdot 4 + 6 = 2 > 0 \checkmark$$

Since this is true for some random value of $x > 3$, we are clear. Therefore, the domain of $f(x)$ is

$$(-\infty, 2) \cup (3, \infty)$$

$$\frac{\sin(\text{gerine})}{\cos(\text{gerine})} =$$



4

Triangles and Circles

4.1 Week 8

1. The minute hand of a clock moves from 12 to 2 o'clock, or $1/6$ of a complete revolution. Through how many degrees does it move? Through how many radians does it move?

Answer: Since there are 360 degrees in a circle, then $1/6$ would be 60 degrees. Equivalently $\pi/3$ radians.

I.e., using $2\pi r$ as the equation of the circumference of a circle:

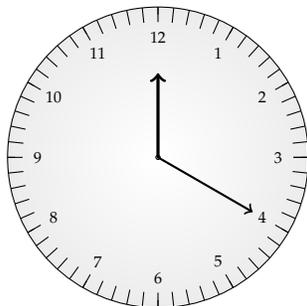
$$s = \frac{1}{6}2\pi r = \frac{\pi}{3}r = r\theta \rightarrow \theta = \frac{\pi}{3}$$

2. The minute hand of a clock is 10 inches long and moves from 12 to 4 o'clock. How far does the tip of the minute hand move? Express your answer in terms of π and then round to two decimal places.

Answer: We can answer this in multiple ways. First, note that to move to the 4, it must move $1/3$ of a circle. The circumference of a circle is $2\pi r$. So if it moves $1/3$ of the whole circle,

$$\text{distance traveled} = \frac{1}{3}2\pi r = \frac{2}{3}\pi \cdot 10$$

because the minute hand is the length of the radius.



3. The figure shows a highway sign that warns of a railway crossing. The lines that form the cross pass through the circle's center and intersect at right angles. If the radius of the circle is 24 inches, find the length of each of the four arcs formed by the cross. Express your answer in terms of π and then round to two decimal places.

Answer:



One way to think about this is the unit circle. We see that because the angles intersect at 90 degrees, each is above the axis as $\pi/4 = 45^\circ$. Then each is arc is $1/4$ the total circumference. Then, that would be $\frac{\pi}{2} \cdot r$, which is

$$\text{length of each arc} = 12\pi$$

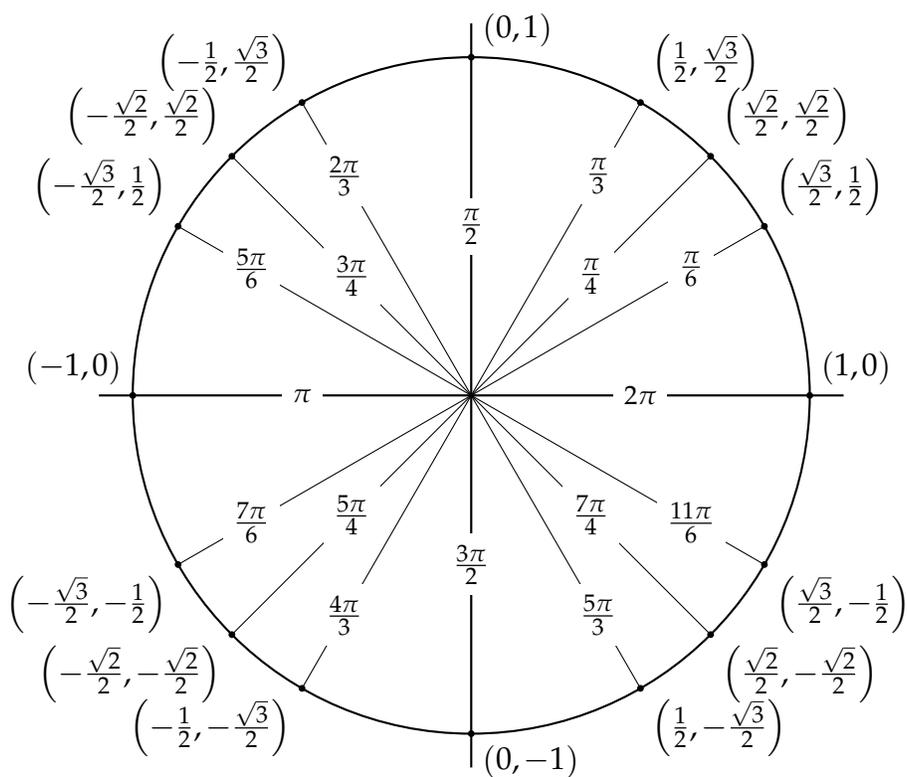


Figure 4.1: Unit circle in radians.

4. How do we measure the distance between two points, A and B , on Earth? We measure along a circle with a center, C , at the center of Earth. The radius of the circle is equal to the distance from C to the surface. Use the fact that the Earth is a sphere of radius equal to approximately 4000 miles to solve the following:

(a) If two points, A and B , are 8000 miles apart, express angle θ in radians.

Answer: So, we know that our distance is 8000 miles. This means we are a circumference times some angle apart of 8000 miles. Remember, $c = 2\pi r\theta$...but, because radians is already in unit of 2π , the 2π 's cancel, and we have: Recall s is arclength, and the equation is $s = r\theta$.

$$\theta = \frac{s}{r} = 2$$

If the 2π part was confusing, imagine we said the distance was $\pi \cdot 4000 = 12566$ miles apart. Then, we'd know we'd be on exact opposite ends of the world, so our angle would be π radians or 180 degrees. Of course, the straight line distance would be just 4000 miles, but since we can't fly through the world, our fastest bet is to travel around the world, and the minimum distance would be 12566 miles. Interesting.

(b) If two points, A and B , are 10,000 miles apart, express angle θ in degrees.

Answer: Here, we know that we have a measurement of 2.5 radians using the c/r equation. Then, we must convert. Note, to convert from radians to degrees, we multiply by $\frac{180}{\pi}$. Then

$$2.5 \text{ radians} \cdot \frac{180}{\pi} \text{ degrees} \approx 143.2 \text{ degrees}$$

(c) If $\theta = 30^\circ$, find the distance between A and B to the nearest mile.

Answer: Now, we backsolve. We know r and θ . Well, almost. Let's convert θ to degrees. Recall, 1 degrees is $\pi/180$ radians, (so 1 radian is $180/\pi$ degrees, which is mostly irrelevant here)

$$30 \text{ degrees} = 30 \cdot \pi/180 = \frac{\pi}{6} \text{ radians}$$

The unit circle is useful there too.

$$s = r\theta = 4000 \cdot \frac{\pi}{6} \approx 666.666\pi \text{ miles}$$

(d) If $\theta = 10^\circ$, find the distance between A and B to the nearest mile.

Answer: This is going to be $1/3$ of the previous answer, so about

$$222.222\pi \text{ miles}$$

4.2 Week 9

1. The seconds hand of a clock is 10.2 centimeters long, as shown in figure 4.2. Assume that 3pm is equivalent to $\theta = 0$ or 2π , i.e. 0 on the the unit circle. What is the value of sine and cosine at 5pm? Use the fact that $\cos^2(x) + \sin^2(y) = r^2$, where r is the radius.

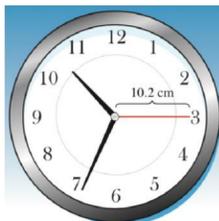


Figure 4.2: Clock

Answer: Reference the unit circle. Moving from 3 to 5 o'clock means we move 60 degrees, which we convert to radians:

$$60^\circ \times \frac{\pi}{180} = \frac{\pi}{3} \text{ radians}$$

However, because we start at 3 o'clock, so we're at $\frac{5\pi}{3}$ on the unit circle, since clocks go clockwise, not counter clockwise like the standard unit circle. Alternatively, we have that we move 300 degrees *counter clockwise*.

$$300^\circ \times \frac{\pi}{180} = \frac{5\pi}{3} \text{ radians}$$

Using our calculators (use $5\pi/3$ if mode is in radians, or 300 degrees if mode is in degrees, check with mode), we see (note $r_x = r \cos(x)$ and $r_y = r \sin(y)$)

$$\cos\left(\frac{5\pi}{3}\right) = \frac{1}{2}r \quad \sin\left(\frac{5\pi}{3}\right) = -\frac{\sqrt{3}}{2}r$$

Remember, $\sin^2(y) + \cos^2(x) = r^2$. Let's check this

$$\sin^2(y) + \cos^2(x) = \left(\frac{1}{2}r\right)^2 + \left(-\frac{\sqrt{3}}{2}r\right)^2 = r^2$$

nice.

2. A sprinkler on a golf course fairway is set to spray water over a distance of 70 feet and rotates through an angle of 120 (see figure 4.3). Find the area of the fairway watered by the sprinkler.

Answer: Essentially, we want the *proportion* of the area of the circle formed by the sprinkler, which sprays radially over time. Remember, the area of a circle is πr^2 .

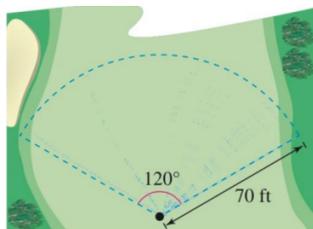


Figure 4.3: Golf

$$\frac{1}{3}\pi r^2 = 5131 \text{ square feet}$$

3. Evaluate the six trigonometric functions at $t = -\frac{\pi}{3}$.

Answer: We're in radians here.

$$\sin\left(-\frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2}$$

$$\cos\left(-\frac{\pi}{3}\right) = \frac{1}{2}$$

$$\tan\left(-\frac{\pi}{3}\right) = -\sqrt{3}$$

$$\csc\left(-\frac{\pi}{3}\right) = -\frac{2}{\sqrt{3}}$$

$$\sec\left(-\frac{\pi}{3}\right) = 2$$

$$\cot\left(-\frac{\pi}{3}\right) = -\frac{1}{\sqrt{3}}$$

4. Evaluate the trigonometric function using its period as an aid.

(a) $\sin\left(\frac{9\pi}{4}\right)$

Answer: Note, $9\pi/4$ is the same as

$$8\pi/4 + \pi/4 = \pi/4$$

because we "reset" every 2π . So,

$$\sin(\pi/4) = \frac{\sqrt{2}}{2}$$

(b) $\cos\left(\frac{8\pi}{3}\right)$

Answer: Note, $8\pi/3$ is the same as $2\pi/3$, because we "reset" every 2π . So,

$$\cos(2\pi/3) = -\frac{1}{2}$$

(c) $\tan\left(\frac{5\pi}{3}\right)$

Answer: We already found $\sin(5\pi/3)$ and $\cos(5\pi/3)$ from question 1. However, the period of tangent is π , not 2π , so we want the $\tan(5\pi/3 - \pi) = \tan(2\pi/3)$.

$$\tan(2\pi/3) = \frac{\sin(2\pi/3)}{\cos(2\pi/3)} = \frac{-\frac{\sqrt{3}}{2}}{1/2} = -\sqrt{3}$$

4.3 Week 10

1. How does the graph of $y = \sin(x)$ compare with the graph of $y = \cos(x)$? Explain how you could horizontally translate the graph of $y = \sin(x)$ to obtain $y = \cos(x)$.

Answer: $\sin(x + \pi/2) = \cos(x)$.

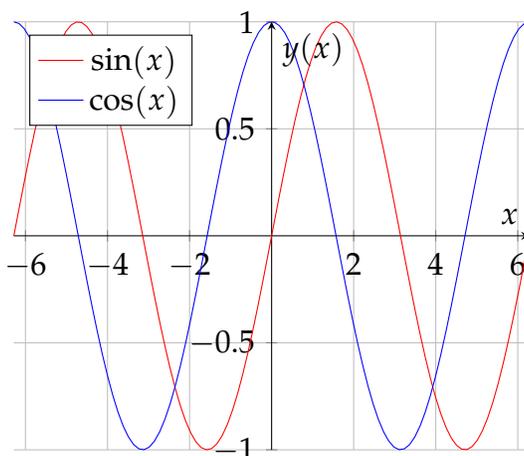


Figure 4.4: Week 10, question 1.

2. State the amplitude, period, and phase shifts of the functions. Draw a graph with labelled axes and at least one full period.

Answer:

(a) $f(x) = 4\cos(\pi x)$

Answer: The amplitude is 4. The period is $2\pi/\pi = 2$, and there is no phase shift.

(b) $f(x) = 2\sin(3x - 21)$

Answer:

The period is $2\pi/3$. The amplitude is 2, and the phase shift is -7. Why, because rewrite

$$f(x) = 2\sin(3x - 21) = 2\sin(3(x - 7))$$

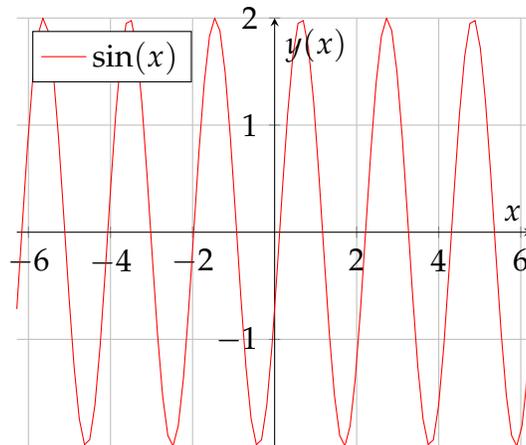
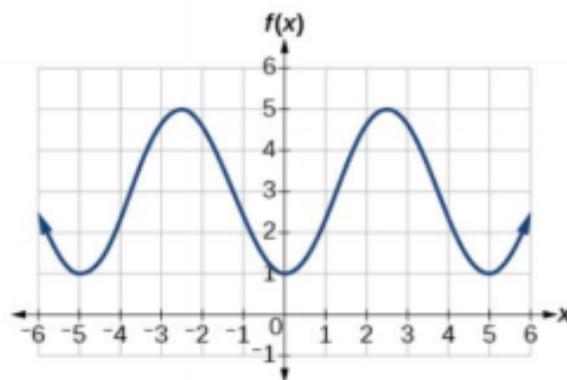


Figure 4.5: Week 10, question 2

3. Determine the amplitude and period of the cosine function from the graph:



Answer: The period is still 5, because of the difference between the maxes. The amplitude is 2, because the difference between top and bottom is 4, as opposed to the usual -1 to 1 .

4. Explain why the period of $y = \tan(x)$ is equal to π

Answer: Well, let's plot it first. In general, the period is the value at which the

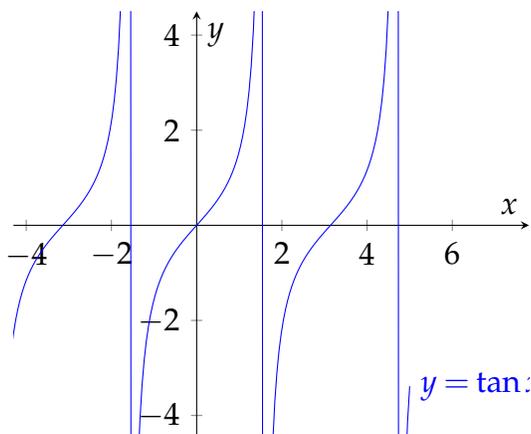
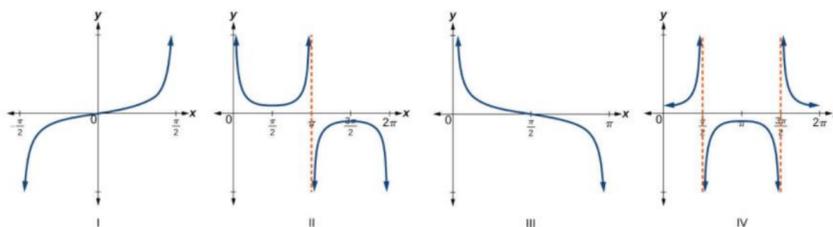


Figure 4.6: Week 10, question 2.

function repeats. It repeats every π .

5. Match the functions $y = \tan(x)$, $y = \sec(x)$, $y = \csc(x)$, and $y = \cot(x)$ with one of the following graphs. Explain how you knew which graph goes with which.



Answer: The first one is the tan graph. Then, the third one is the cot graph, because we see it is the reciprocal. The second graph is csc, which we can see by plotting, and then the fourth is sec. Additionally, the asymptotes tell the story. $\sin(\pi) = 0$ so $1/\sin(\pi)$ will be an asymptote.

4.4 Week 11

1. For what domain can we take the inverse of sine, cosine, and tangent? Why do we need to restrict the domain in order to find the inverse?

Answer: The inverse function tells us what number would return the argument as the output. I.e. $\sin(\pi/2) = 1$, so what is $\sin^{-1}(1) = \pi/2$. Therefore, the domain of the \sin^{-1} and \cos^{-1} , aka the arcsin and arccos is $[-1,1]$. For the domain of the

inverse tangent, this is $(-\infty, \infty)$, because we can plot it and see this. The range of $\tan^{-1} \in (-\pi/2, \pi/2)$.

2. Evaluate the following. Make sure to check if the inner value is within the domain.

(a) $\sin^{-1}(\sin(\frac{\pi}{3}))$

Answer: Here, we want to find angle the sine argument would return the value $\sin(\pi/3) = \frac{\sqrt{3}}{2}$. Of course, this occurs when $\theta = \pi/3$.

$$\sin^{-1}(\sin(\frac{\pi}{3})) = \frac{\pi}{3}$$

(b) $\sin^{-1}(\sin(\frac{2\pi}{3}))$

Answer: Here, we want to find angle the sine argument would return the value $\sin(2\pi/3) = \frac{\sqrt{3}}{2}$. So this is the same as part (a).

$$\sin^{-1}(\sin(\frac{2\pi}{3})) = \frac{\pi}{3}$$

(c) $\cos^{-1}(\cos(\frac{2\pi}{3}))$

Answer: The $\cos(2\pi/3) = -1/2$. So what angle θ does $\cos(\theta) = -1/2$.

$$\cos^{-1}(\cos(\frac{2\pi}{3})) = \frac{2\pi}{3}$$

(d) $\cos^{-1}(\cos(\frac{-\pi}{3})) = \cos^{-1}(\cos(\frac{5\pi}{3}))$

Answer: Notice, $\cos(-\pi/3) = \cos(5\pi/3) = 1/2$, which occurs at $\cos(\pi/3)$ as well, so

$$\cos^{-1}(\cos(\frac{5\pi}{3})) = \frac{\pi}{3}$$

which will repeat every π .

3. Find the angle θ . Use a calculator and round to two decimal places.

Answer: Well, we are given the opposite angle and the adjacent, but not the hypotenuse, which is $h = \sqrt{12^2 + 19^2} = 22.47$. Not that that is relevant. Anyways, this is exactly why we introduce the inverse. Typically, we are *given* the angle, and one of the side lengths, and we solve for the other. Here, we know what the tangent should equal $12/19$. That does not mean $\theta = \tan(12/19)$, because we define tangent as the angle that gives $12/19$, not the other way. With that in mind, what angle θ will give us the tan equal to $12/19$. Using the inverse tangent function on the calculator, we find that

$$\tan^{-1}(12/19) = 0.563 \text{ radians} = 32.28 \text{ degrees}$$

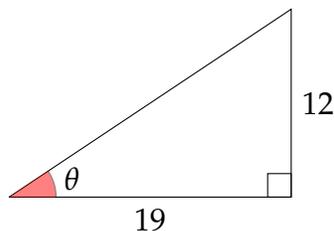


Figure 4.7: Not to scale

4. Suppose you drive 0.6 miles on a road so that the vertical distance changes from 0 to 150 feet. What is the angle of elevation of the road? (Hint: It might help to draw a picture. Additionally, make sure that the units of your distances are the same before you input into an inverse trig function).

Answer: This is similar to question 4, except now we now the opposite side and



Figure 4.8: Not to scale

the hypotenuse. We know that the sine of our angle equals $150/3168$, after we convert 0.6 miles to 3168 feet. We then need to use the inverse sine function to find our angle.

$$\sin^{-1}(150/3168) = 0.047 \text{ radians} = 2.71 \text{ degrees}$$

5. Suppose that $\sin(t) = \frac{x}{x+1}$. Find

(a) $\cos(t)$

Answer: We first must find t . Notice,

$$t = \sin^{-1}\left(\frac{x}{x+1}\right)$$

So, we are looking for

$$\cos\left(\sin^{-1}\left(\frac{x}{x+1}\right)\right)$$

Remember the equation (from $\sin^2(x) + \cos^2(x) = 1$),

$$\cos(x) = \sqrt{1 - \sin^2(x)}$$

So, we have that

$$\cos(t) = \cos\left(\sin^{-1}\left(\frac{x}{x+1}\right)\right) = \sqrt{1 - \sin^2\left(\sin^{-1}\left(\frac{x}{x+1}\right)\right)}$$

However, note that $\sin(\sin^{-1}(x)) = x$, in general. Also, it is important to know that $\sin^2(x)$ is simply squaring the sin function. That means, we have that, in general, $\sin^2(x) = \sin(x) \cdot \sin(x)$. Notice,

$$\sin\left(\sin^{-1}\left(\frac{x}{x+1}\right)\right) \cdot \sin\left(\sin^{-1}\left(\frac{x}{x+1}\right)\right) = \frac{x}{x+1} \cdot \frac{x}{x+1}$$

so

$$\sin^2\left(\sin^{-1}\left(\frac{x}{x+1}\right)\right) = \left(\frac{x}{x+1}\right)^2$$

and we have

$$\cos(t) = \sqrt{1 - \frac{x^2}{(x+1)^2}}$$

Alternatively, we can use the Pythagorean formula. Because $\sin(t) = \frac{x}{x+1}$, then we know our triangles opposite is x and the hypotenuse is $x+1$. Then the adjacent angle, y , is

$$\begin{aligned} y^2 + x^2 &= (x+1)^2 \\ y^2 &= (x+1)^2 - x^2 \\ y &= \sqrt{(x+1)^2 - x^2} \end{aligned}$$

Then $\cos(t) = \frac{\sqrt{(x+1)^2 - x^2}}{x+1}$. We can simplify this by noting that $(x+1) = \sqrt{(x+1)^2}$. With this in mind, we get the same answer as before.

Alternatively, the answer is (if we do $(x+1)^2 - x^2 = 2x+1$),

$$\cos(t) = \frac{\sqrt{2x+1}}{x+1}$$

(b) $\cot(t)$

Answer: We sort of use the same steps as in part (a). Remember, (using t here because this will directly lead to the answer)

$$\cot(t) = \frac{\cos(t)}{\sin(t)}$$

So, using the result from part a,

$$\cot(t) = \frac{\cos(t)}{\sin(t)} = \frac{\sqrt{1 - \frac{x^2}{(x+1)^2}}}{x/(x+1)} = \frac{(x+1)\sqrt{1 - \frac{x^2}{(x+1)^2}}}{x}$$

Which is also using the alternative from (a):

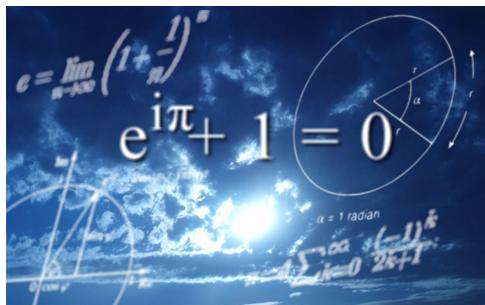
$$\frac{\sqrt{2x+1}}{x}$$

(c) $\cos\left(\sin^{-1}\left(\frac{x}{x+1}\right)\right)$

Answer: This is the same as part (a). Not sure why that is there.

This equation is supposedly the most beautiful of all time.

Eulers formula



5

Angle Identities

5.1 Week 12

1. Verify the following identities:

(a) $\cos(x) - \cos^3(x) = \cos(x) \sin^2(x)$

Answer:

$$\begin{aligned} \cos(x) - \cos^3(x) &= \cos(x) - \cos(x) \cos^2(x) \\ &= \cos(x) \cdot (1 - \cos^2(x)) \\ &= \cos(x) \sin^2(x) \checkmark \end{aligned}$$

(b) $(\sin(x) + \cos(x))^2 = 1 + 2 \sin(x) \cos(x)$

Answer:

$$\begin{aligned} (\sin(x) + \cos(x))^2 &= \underbrace{\sin^2(x) + \cos^2(x)}_1 + 2 \sin(x) \cos(x) \\ &= 1 + 2 \sin(x) \cos(x) \checkmark \end{aligned}$$

(c) $\sin(x + y) + \sin(x - y) = 2 \sin(x) \cos(y)$

Answer: Recall,

$$\sin(x + y) = \sin(x) \cos(y) + \sin(y) \cos(x) \tag{5.1}$$

$$\sin(x - y) = \sin(x) \cos(y) - \sin(y) \cos(x) \tag{5.2}$$

Then, we have

$$\begin{aligned}\sin(x + y) + \sin(x - y) &= \sin(x)\cos(y) + \sin(y)\cos(x) + \sin(x)\cos(y) - \sin(y)\cos(x) \\ &= 2\sin(x)\cos(y)\checkmark\end{aligned}$$

(d) $\cos(x + y)\cos(x - y) = \cos^2(x) - \sin^2(y)$

Answer: Here, we begin by noting that:

$$\cos(x + y) = \cos(x)\cos(y) - \sin(x)\sin(y) \quad (5.3)$$

$$\cos(x - y) = \cos(x)\cos(y) + \sin(x)\sin(y) \quad (5.4)$$

Which means

$$\begin{aligned}\cos(x + y)\cos(x - y) &= \left(\underbrace{\cos(x)\cos(y)}_a - \underbrace{\sin(x)\sin(y)}_b \right) \cdot \left(\cos(x)\cos(y) + \sin(x)\sin(y) \right) \\ &= \cos^2(x)\cos^2(y) - \sin^2(x)\sin^2(y) \\ &= \cos^2(x)(1 - \sin^2(y)) - (1 - \cos^2(x))\sin^2(y) \\ &= \cos^2(x) - \cos^2(x)\sin^2(y) - \sin^2(y) + \cos^2(x)\sin^2(y) \\ &= \cos^2(x) - \sin^2(y)\checkmark\end{aligned}$$

(Using difference of squares $(a - b)(a + b) = a^2 - b^2$).

Aside If you ever forget some of the trig-identities, here is a good way to remember:

Fun Trick (don't need to know this)

- If you ever forget some of the identities, here's a trick: Euler's formula:

$$e^{ix} = \cos(x) + i \sin(x)$$

- An example of how this is helpful:

$$e^{i(x+y)} = e^{ix} e^{iy}$$

$$\begin{aligned} \xrightarrow{\text{Euler's formula}} &= \cos(x+y) + i \sin(x+y) = (\cos(x) + i \sin(x)) (\cos(y) + i \sin(y)) \\ &= \cos(x) \cos(y) + i \cos(x) \sin(y) + i \sin(x) \cos(y) + i^2 \sin(x) \sin(y) \end{aligned}$$

$$\begin{aligned} \xrightarrow{\text{which means}} &= \cos(x+y) + i \sin(x+y) \\ &= (\cos(x) \cos(y) - \sin(x) \sin(y)) + i (\cos(x) \sin(y) + \cos(y) \sin(x)) \end{aligned}$$

If we equate the terms with imaginary numbers attached to them and those without, we see:

$$\begin{aligned} \cos(x+y) &= \cos(x) \cos(y) - \sin(x) \sin(y) \\ \sin(x+y) &= \cos(x) \sin(y) + \cos(y) \sin(x) \end{aligned}$$

2. Given $\sin(a) = \frac{4}{5}$ and $\cos(b) = \frac{1}{3}$ with a and b both in the interval $[0, \frac{\pi}{2})$, find $\sin(a-b)$ and $\cos(a+b)$.

Answer: So $a = \sin^{-1}(\frac{4}{5}) = 0.927$ and $b = \cos^{-1}(\frac{1}{3}) = 1.23$ (both in radian scale). So, just so we know we get the right answer ahead of time, we check our calculators and find $\sin(a-b) = \sin(-0.303) = -0.299$ radians and $\cos(a+b) = \cos(2.157) = -0.553$ radians. However, we'd like to confirm this analytically, which we can do using our trig identities from equations (5.2) and (5.4). We know that $\sin(\cos^{-1}(x)) = \sqrt{1-x^2}$ and $\cos(\sin^{-1}(x)) = \sqrt{1-x^2}$. How do we know this?

Well, let $\cos^{-1}(x) = \theta$, some angle, so then $x = \cos(\theta)$. If we draw a triangle in a unit circle with hypotenuse 1, and adjacent side x , then we see that opposite side length is $\sqrt{1-x^2}$ from the Pythagorean theorem. See figure 5.1.

We can do the same reasoning to get the $\cos(\sin^{-1}(x)) = \sqrt{1-x^2}$ relationship, try for yourself if you'd like.

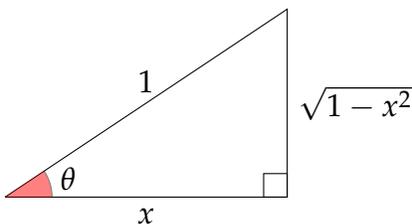


Figure 5.1: In our example, the side lengths would be a or b not x .

$$\sin(a - b) = \sin(a) \cos(b) - \sin(b) \cos(a) = \frac{4}{5} \cdot \frac{1}{3} - \sqrt{1 - \left(\frac{1}{3}\right)^2} \cdot \sqrt{1 - \left(\frac{4}{5}\right)^2} = -.299 \text{ radians}$$

as expected. Alternatively, from the triangle $\cos(a) = \frac{3}{5}$ and $\sin(b) = \frac{\sqrt{8}}{3}$, using the Pythagorean theorem. Similarly,

$$\cos(a + b) = \cos(a) \cos(b) - \sin(a) \sin(b) = \sqrt{1 - \left(\frac{4}{5}\right)^2} \cdot \frac{1}{3} - \frac{4}{5} \cdot \sqrt{1 - \left(\frac{1}{3}\right)^2} = -0.554 \text{ radians}$$

Alternatively, drawing out the triangle for b and using the Pythagorean theorem, we have

$$\sin a \cos b - \cos a \sin b = \frac{4}{5} \frac{1}{3} - \frac{3}{5} \frac{2\sqrt{2}}{3} = \frac{4 - 6\sqrt{2}}{15}$$

- 3. Find the exact value of $\sin\left(\cos^{-1}(0) - \cos^{-1}(1/2)\right)$.**

Answer: Note, $\cos^{-1}(0) = \frac{\pi}{2}$ and $\cos^{-1}(1/2) = \frac{\pi}{3}$. Therefore, we need

$$\sin(\pi/6) = 1/2$$

using the unit circle.

- 4. Set up the solution of $\cos\left(\frac{5\pi}{4}\right)$ in two different ways using the sum and difference identities for cosine, and then compute them to make sure they give the same answer.**

Answer:

$$\cos\left(\frac{5\pi}{4}\right) = \cos(\pi + \pi/4) = \cos(\pi) \cos(\pi/4) - \underbrace{\sin(\pi) \sin(\pi/4)}_{=0} = -\cos(\pi/4) = \cos(5\pi/4) \checkmark$$

$$\cos\left(\frac{5\pi}{4}\right) = \cos\left(2\pi - \frac{3\pi}{4}\right) = \cos(2\pi) \cos\left(\frac{3\pi}{4}\right) + \underbrace{\sin(2\pi) \sin\left(\frac{3\pi}{4}\right)}_{=0} = \cos\left(\frac{3\pi}{4}\right) = \cos\left(\frac{5\pi}{4}\right) \checkmark$$

5.2 Week 13

1. Use the half-angle relationships to solve the following.

(a) Find the value of $\sin(\pi/8)$.

Answer: The following identities may be useful here: The double angle formulas

$$\sin(2x) = 2\sin(x)\cos(x) \quad (5.5)$$

$$\cos(2x) = 1 - 2\sin^2(x) \quad (5.6)$$

$$\tan(2x) = \frac{2\tan(x)}{1 - \tan^2(x)} \quad (5.7)$$

The power-reducing formulas

$$\sin^2(x) = \frac{1 - \cos(2x)}{2} \quad (5.8)$$

$$\cos^2(x) = \frac{1 + \cos(2x)}{2} \quad (5.9)$$

$$\tan^2(x) = \frac{1 - \cos(2x)}{1 + \cos(2x)} \quad (5.10)$$

and finally the half angle formulas

$$\sin\left(\frac{x}{2}\right) = \pm\sqrt{\frac{1 - \cos(x)}{2}} \quad (5.11)$$

$$\cos\left(\frac{x}{2}\right) = \pm\sqrt{\frac{1 + \cos(x)}{2}} \quad (5.12)$$

$$\tan\left(\frac{x}{2}\right) = \pm\sqrt{\frac{1 - \cos(x)}{1 + \cos(x)}} \quad (5.13)$$

Note, using the half angle identity (formula), $x/2 = \pi/8$, so $x = \pi/4$. Then, since $\pi/4$ is in quadrant 1,

$$\sin(\pi/8) = \sqrt{\frac{1 - \cos(\pi/4)}{2}} = \sqrt{\frac{1 - \sqrt{2}/2}{2}} = \boxed{\frac{\sqrt{2} - \sqrt{2}}{2}}$$

We keep the positive square root because we're in quadrant 1.

(b) Find the value of $\cos(\pi/8)$.

Answer: This is the same as (a) but with a plus in the numerator.

$$\boxed{\cos(\pi/8) = \frac{\sqrt{2} + \sqrt{2}}{2}}$$

(c) Find the value of $\tan(\pi/12)$.

Answer: Again, let $x/2 = \pi/12$ so $x = \pi/6$. We choose the positive square root because $\pi/12$ is in quadrant 1. Then using the half angle formula:

$$\tan\left(\frac{\pi}{12}\right) = \sqrt{\frac{1 - \sqrt{3}/2}{1 + \sqrt{3}/2}} = 2 - \sqrt{3}$$

2. Evaluate $\sin(\pi/12)$ and $\cos(\pi/12)$ exactly.

Answer: Again, let $x/2 = \pi/12$ so $x = \pi/6$. We choose the positive square root because $\pi/12$ is in quadrant 1. Then

$$\begin{aligned} \sin(x/2) &= \sin\left(\frac{\pi}{12}\right) = \sqrt{\frac{1 - \cos(\pi/6)}{2}} \\ &= \sqrt{\frac{1 - \sqrt{3}/2}{2}} \\ &= \boxed{\frac{\sqrt{2 - \sqrt{3}}}{2}} \end{aligned}$$

And because the half-angle identity for \cos differs only by a plus sign in the numerator:

$$\cos\left(\frac{\pi}{12}\right) = \boxed{\frac{\sqrt{2 + \sqrt{3}}}{2}}$$

The ratio should give us the same answer as we got in 1(c). Check it if you'd like.

3. Confirm the following relationships by showing the left hand side equals the right hand side.

(a) $2\sin^2(x/2) + \cos(x) = 1$

Answer: Recall that $\sin^2(x/2) = \sin(x/2) \cdot \sin(x/2)$.

$$\begin{aligned} 2\sin^2(x/2) + \cos(x) &= 2\left(\frac{1 - \cos(x)}{2}\right) + \cos(x) \\ &= 1 - \cos(x) + \cos(x) \\ &= 1\checkmark \end{aligned}$$

(b) $2\cos^2(x/2)\tan(x) = \tan(x) + \sin(x)$

Answer:

$$\begin{aligned} 2\cos^2(x/2)\tan(x) &= 2\left(\frac{1 + \cos(x)}{2}\right)\tan(x) \\ &= (1 + \cos(x))\tan(x) \\ &= \tan(x) + \cos(x)\frac{\sin(x)}{\cos(x)} \\ &= \tan(x) + \sin(x)\checkmark \end{aligned}$$

4. $\cos(\theta) = \sqrt{3}/7$, and $0 \leq \theta \leq \pi/2$. Find $\sin(\theta/2)$ and $\tan(\theta/2)$.

Answer: Since we know $\cos(\theta)$, we can use the half-angle identities to solve for $\sin(\theta/2)$ and $\cos(\theta/2)$.

First, we solve for $\sin(\theta/2)$ as follows:

$$\sin(\theta/2) = \sqrt{\frac{1 - \sqrt{3}/7}{2}} = \sqrt{\frac{7 - \sqrt{3}}{14}}$$

Which means, using our calculators to confirm:

$$\theta = 2 \sin^{-1} \left(\sqrt{\frac{7 - \sqrt{3}}{14}} \right) = \cos^{-1} \left(\sqrt{3}/7 \right) \checkmark$$

$$\tan(\theta/2) = \sqrt{\frac{1 - \sqrt{3}/7}{1 + \sqrt{3}/7}} = \sqrt{\frac{7 - \sqrt{3}}{7 + \sqrt{3}}} = \sqrt{\frac{(7 - \sqrt{3})^2}{46}}$$

where in the last step we multiplied by $\frac{7 - \sqrt{3}}{7 - \sqrt{3}}$ to simplify the denominator.

5. Solve $\tan(2x) + \tan(x) = 0$ for $0 \leq x \leq 2\pi$.

Answer: We will use the double angle formula and factoring in this question.

$$\begin{aligned} \tan(2x) + \tan(x) &= 0 \\ \xrightarrow{\text{Double-angle formula}} \frac{2 \tan(x)}{1 - \tan^2(x)} + \tan(x) &= 0 \\ 2 \tan(x) + \tan(x) (1 - \tan^2(x)) &= 0 \\ 2 \tan(x) + \tan(x) - \tan^3(x) &= 0 \\ 3 \tan(x) - \tan^3(x) &= 0 \\ \tan(x) (3 - \tan^2(x)) &= 0 \end{aligned}$$

We must now factor our two terms. This is because this equals zero only when one of the terms equals zero, i.e. what arguments of x make this statement true, so we solve each equal to 0. The term on the left is pretty simple, as

$$\tan(x) = 0 \implies x = 0, \pi$$

On the right, we have

$$3 - \tan^2(x) = 0 \implies \tan(x) = \pm\sqrt{3} \implies x = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$$

where the last step can be confirmed by graphing or by looking at our unit circle.

5.3 Week 14

1. Find all angles that make the following equations true:

(a) $\csc(x) = 2$.

Answer: True when $\sin(x) = \frac{1}{2}$. This is seen on the unit circle at $x = \pi/6$ and $5\pi/6$. However, we want all solutions, so since this repeats every 2π , we have

$$x = \frac{\pi}{6} \pm 2\pi n, \frac{5\pi}{6} \pm 2\pi n$$

(b) $\sec(x) = 2/\sqrt{2}$.

Answer: This question is similar to last. We wanna find when $\cos(x) = \sqrt{2}/2$. Using the unit circle, we see

$$x = \frac{\pi}{4} \pm 2n\pi, \frac{7\pi}{4} \pm 2n\pi$$

2. Solve the equation for $x \in [0, 2\pi)$: $2\cos^2(2x) - \cos(2x) - 1 = 0$.

Answer: Let $t = \cos(2x)$. Then we have

$$2t^2 - t - 1 = 0 \xrightarrow{\text{factor}} (t - 1)(2t + 1)$$

If we plug back in $\cos(2x) = t$, then we have

$$(\cos(2x) - 1)(2\cos(2x) + 1) = 0$$

This means by the zero product property, since one of them (at least) must equal zero,

$$\cos(2x) - 1 = 0$$

$$2\cos(2x) + 1 = 0$$

Recall, $\cos(2x) = 1 - 2\sin^2(x)$, so the first term is

$$\cos(2x) = 1 \implies 1 - 2\sin^2(x) = 1 \implies \sin^2(x) = 0$$

which means $x = \pi n$ is a solution, because $\sin(x) = 0$ at $0, \pi, 2\pi, \dots$. However, we are restricted by the domain in the question, so we just keep $x = 0, \pi$. The second term says

$$\cos(2x) = -\frac{1}{2}$$

$$1 - 2\sin^2(x) = -\frac{1}{2}$$

$$\sin^2(x) = \frac{3}{4}$$

$$\sin(x) = \pm \frac{\sqrt{3}}{2}$$

$$x = \frac{\pi}{3}, \frac{2\pi}{3} \text{ and } \frac{4\pi}{3}, \frac{5\pi}{3}$$

Thus,

$$x = 0, \pi, \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$$

3. Calculate the length of c

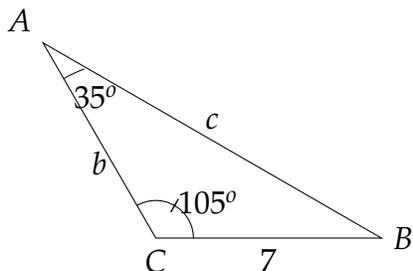


Figure 5.2

Answer: We use the law of sines here. Recall,

$$\text{Law of Sines: } \frac{\sin(A)}{a} = \frac{\sin(B)}{b} = \frac{\sin(C)}{c} \quad (5.14)$$

In our example,

$$\frac{7}{\sin(35)} = \frac{c}{\sin(105)} \implies \boxed{c = 11.8}$$

4. Find all the unknown angles and side lengths given $a = 1$, $b = 3$ and $c = 2.4$.

Answer: We use the law of cosines, that

$$a^2 = b^2 + c^2 - 2bc \cos(A) \quad (5.15)$$

Let's plug all that in

$$\begin{aligned} 1^2 &= 3^2 + 2.4^2 - 2 \cdot 3 \cdot 2.4 \cos(A) \\ -13.76 &= -14.4 \cos(A) \\ 0.955 &= \cos(A) \\ A &= \cos^{-1}(0.955) = 0.30 \text{ radians} = 17.25^\circ \end{aligned}$$

However, we are not done. We know the angle A , but we still want angle B and C . Then, we have from the law of sines that

$$\frac{\sin(17.25)}{1} = \frac{\sin(C)}{2.4} = 45.37^\circ$$

and then the angle B equals $180 - 17.25 - 45.37 = 117.30^\circ$. Solving for B first requires you to check that there is not a solution in the second quadrant, i.e. $180 - \text{the angle you'll get by plugging in } B$ in the law of sines. If you get $\angle B < \angle C$, you know you have an issue because $b < c$! Therefore, that's how we know we have to check to see if we are in the correct quadrant.

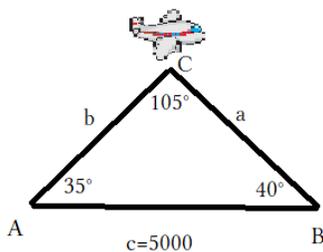


Figure 5.3: The triangle from question 4 of the last hw.

5.4 Final Exam Review

1. Find the domain:

(a) $f(x) = \frac{1-x}{x^2-9}$.

Answer: Factor the denominator and get

$$\frac{1-x}{(x+3)(x-3)}$$

That means we have asymptotes at $x = -3$ and $x = 3$, so the domain is

$$(-\infty, -3) \cup (-3, 3) \cup (3, \infty)$$

(b) $f(x) = \frac{x+1}{3x-2}$.

Answer: The denominator is easy to factor, after setting $3x - 2 = 0$, because we cannot divide by zero, we have domain

$$(-\infty, 2/3) \cup (2/3, \infty)$$

(c) $f(x) = \ln(2x + 1)$.

Answer: Again, the log function cannot take a negative argument, so the domain is

$$(-1/2, \infty)$$

(d) $f(x) = \sqrt{8 - 2x}$

Answer: The square root cannot be negative. However, since we are multiplying by -2 , this means the domain is

$$(-\infty, 4]$$

The right bracket is closed because the $\sqrt{0}$ exists.

2. Evaluate the difference quotient:

(a) $-\frac{1}{3x}$

Answer: The difference quotient is

$$\frac{f(x+h) - f(x)}{h}$$

We know $f(x)$, but we need to calculate $f(x+h)$.

$$\frac{-\frac{1}{3(x+h)} + \frac{1}{3x}}{h} = \frac{1}{3x(h+x)}$$

(b) $-x^2 + 5x + 9$

Answer:

$$\frac{(-(h+x)^2 + 5(h+x) + 9) - (-x^2 + 5x + 9)}{h} = -h - 2x + 5$$

(c) $3x^2 + 4x - 8$

Answer:

$$\frac{(3(h+x)^2 + 4(h+x) - 8) - (3x^2 + 4x - 8)}{h} = 3h + 6x + 4$$

3. Find the function $g(x)$ after applying the following transformations to x^2 . reflect about the x -axis, shift left 5 units, shift up 3 units.

Answer: Reflecting means we multiply the argument by negative 1, a shift to the left means the function argument has 5 *added* to it, and the shift up of 3 means we add 3 outside the argument.

4. Find compositions:

(a) Find $(f \circ g)(x)$ and $(g \circ f)(x)$ where $f(x) = x^2 - x + 4$ and $g(x) = 2x - 3$.

Answer:

$$(f \circ g)(x) = (2x - 3)^2 - (2x - 3) + 4 = 4x^2 - 14x + 16$$

$$(g \circ f)(x) = 2(x^2 - x + 4) - 3 = 2x^2 - 2x + 5$$

(b) Find $(g \circ f)(x)$ where $f(x) = e^{2x} - 1$ and $g(x) = \ln(x + 1)$.

Answer:

$$(g \circ f)(x) = \ln(e^{2x} - 1 + 1) = 2x$$

5. Find inverse functions

(a) $f(x) = y = \frac{4x}{3+x}$

Answer: Swap x and y and then solve for y

$$x = \frac{4y}{3+y} \rightarrow y = \frac{-3x}{x-4} = f^{-1}(x)$$

(b) $f(x) = y = x^3 - 10$

Answer: Same procedure as before, we have $x + 10 = y^3$, which means

$$f^{-1}(x) = (x + 10)^{1/3}$$

6. **An astronaut on the moon throws a baseball upward. The height of the ball is approximated by the function $h(t) = -2.7t^2 + 30t + 6.5$ feet, t is time in seconds after the ball was thrown. When does the ball reach its maximum height? What is the maximum height of the baseball?**

Answer: Notice, this equation is different because the strength of gravity is different on the moon! Because this is a quadratic, we can find the vertex from $-b/2a$, to get t -coordinate, plug back in at t to get h . Therefore, the vertex occurs at $-30/(-5.4)$ which is at $t \approx 5.555$ seconds which means the height is $h \approx 89.8$ feet. We know this a max because the leading coefficient on t^2 is negative meaning we are slowing down, meaning the initial speed is faster, meaning the ball will eventually slow down to the point that it'll turn around and fall.

7. Find all zeros

(a) $f(x) = -x^3 + x^2 + 2x$

Answer:

$$x(-x^2 + x + 2) = x(x-1)(x+2) \implies x=0, x=1, x=-2$$

(b) $f(x) = x^2 - x^2 + 9x - 9$

Answer: When we see a cube and can't simply factor out a x , we have to work a bit. We can factor this into

$$f(x) = (x-1)(x^2+9) \implies x=1, x=\pm 3i$$

so this is an example with imaginary roots.

8. **Suppose that the insect population in millions is modeled by $f(x) = \frac{10x+1}{0.2x+1}$, where $x \geq 0$ is in months. What happens to the insect population after a long time?**

Answer: If x gets big, the 1 doesn't matter, and the ratio $10/0.2$ is all that matters. So the population stagnates at 50 million eventually.

9. **A company that manufactures calculators has determined that the average cost for producing x calculators is $\bar{C} = \frac{15000+20x}{x}$ dollars. In the long run, what value does the average cost approach?**

Answer: This is the same thinking as the previous question. Don't be fooled by the 15,000. Over enough time, even this big number is dominated by x , so the average cost to manufacture approaches 20 dollars.

10. Solve exponential and logarithmic equations

(a) $3^{2x} - 3^x - 42 = 0$

Answer: we rewrite as

$$(3^x)^2 - 3^x - 42 = 0$$

let $u = 3^x$, then we factor $u^2 - u - 42 = 0$ meaning $u = 7, u = -6$. Then

$$3^x = 7 \implies x = \frac{\ln(7)}{\ln(3)}$$

$$3^x = -6 \implies \text{no solution}$$

(b) $5^x = 3^{x-1}$

Answer: Take the \ln of both sides:

$$x \ln(5) = (x - 1) \ln(3) \implies x = -\frac{\ln(3)}{\ln(5) - \ln(3)}$$

(c) $\log_2(x) + \log_2(x - 7) = 3$

Answer: We exponentiate this time with 2 as the base: (recall $\ln(a + b) = \ln(a) \ln(b)$)

$$x(x - 7) = 2^3 \implies x = 8, -1$$

but the $x = -1$ solution does not exist in the domain $(0, \infty)$.

(d) $\ln(x) - \ln(x - 2) = 1$

Answer: Same situation as before but with e instead of 2, and because $\ln(a - b) = \ln(a) / \ln(b)$

$$x/(x - 2) = e \implies x = e(x - 2)$$

Now we have to solve for x .

$$ex - 2e = x$$

$$ex - x - 2e = 0$$

$$x(e - 1) = 2e$$

$$x = \frac{2e}{e - 1}$$

11. Find the amplitude, period, and phase shift for the function $y = -3 \cos(2x + \pi)$.

Answer: The amplitude is 3 (the direction doesn't matter, its just a magnitude), the period is $2\pi/2 = \pi$, and the phase shift is, after factoring the 2 out, $-3 \cos(2(x + \pi/2))$, which means we shift $-\pi/2$, or $\pi/2$ to the left.

12. Verify the trigonometric identities

(a) $\cos(x) \cot(x) + \sin(x) = \csc(x)$

Answer:

$$\begin{aligned} \cos(x) \cot(x) + \sin(x) &= \cos(x) \frac{\cos(x)}{\sin(x)} + \sin(x) \\ &= \frac{\cos^2(x)}{\sin(x)} + \frac{\sin^2(x)}{\sin(x)} \\ &= \frac{\cos^2(x) + \sin^2(x)}{\sin(x)} \\ &= \frac{1}{\sin(x)} = \csc(x) \checkmark \end{aligned}$$

(b) $\frac{\cos(x) + \sin(x) - \sin^3(x)}{\sin(x)}$

Answer: Make sure to have your trig identities handy

$$\begin{aligned} \frac{\cos(x) + \sin(x) - \sin^3(x)}{\sin(x)} &= \frac{\cos(x)}{\sin(x)} + \frac{\sin(x)}{\sin(x)} - \frac{\sin^3(x)}{\sin(x)} \\ &= \cot(x) + \frac{\sin(x) - \sin^3(x)}{\sin(x)} \\ &= \cot(x) + \frac{\sin(x)(1 - \sin^2(x))}{\sin(x)} \\ &= \cot(x) + 1 - \sin^2(x) \\ &= \cot(x) + \cos^2(x) \checkmark \end{aligned}$$

13. Given $\sin(\alpha) = -\frac{3}{8}$ for $\pi < \alpha < \frac{3\pi}{2}$ and $\cos(\beta) = \frac{3}{5}$ for $0 < \beta < \frac{\pi}{2}$, find the following

(a) $\cos(\alpha)$

Answer: For the α triangle, we back-solve with the Pythagorean theorem. If we know $\sin(\alpha)$, then we know the opposite and hypotenuse lengths. Then the adjacent length, call it a is

$$a = \sqrt{8^2 - (-3)^2} = \pm\sqrt{55}$$

Then

$$\cos(\alpha) = -\frac{\sqrt{55}}{8}$$

because of the restriction that $\alpha \in (\pi, 3\pi/2)$, which is in quadrant 3, we know we must take the negative of the square root.

(b) $\sec(\alpha), \tan(\alpha), \cot(\alpha), \csc(\alpha)$

Answer: These are now plug and chug

$$\sec(\alpha) = -\frac{8\sqrt{55}}{55} \tan(\alpha) = \frac{3\sqrt{55}}{55} \cot(\alpha) = \frac{\sqrt{55}}{3} \csc(\alpha) = -\frac{8}{3}$$

(c) $\cos(2\alpha), \sin(2\alpha), \tan(2\alpha)$

Answer: Use the identities that $\cos(2x) = 1 - 2\sin^2(x)$, $\sin(2x) = 2\sin(x)\cos(x)$, and $\tan(2x) = \frac{2\tan(x)}{1-\tan^2(x)}$, which yields

$$\cos(2\alpha) = 1 - 2\left(\frac{-3}{8}\right)^2 = \frac{23}{32}$$

$$\sin(2\alpha) = \frac{3\sqrt{55}}{32}$$

$$\tan(2\alpha) = \frac{3\sqrt{55}}{23}$$

(d) $\cos(\alpha/2)$ and $\sin(\alpha/2)$.

Answer: Use $\cos(x/2) = \pm\sqrt{\frac{1+\cos(x)}{2}}$ and $\sin(x/2) = \sqrt{\frac{1-\cos(x)}{2}}$. Then, we have that (again taking the appropriate square root given the domain of α)

$$\cos(\alpha/2) = -\frac{\sqrt{8-\sqrt{55}}}{4}$$

$$\sin(\alpha/2) = \frac{\sqrt{8+\sqrt{55}}}{4}$$

(e) $\sin(\alpha - \beta)$

Answer: First, we must fill out the β triangle. Because the adjacent side is 3 and hypotenuse is 5, the opposite will be 4, because that fills out the 3-4-5 right triangle. This is convenient. Then we have $\sin(\beta) = 4/5$. Here, we use

$$\sin(x - y) = \sin(x)\cos(y) - \cos(x)\sin(y)$$

for us, this means

$$\begin{aligned} \sin(\alpha - \beta) &= \sin(\alpha)\cos(\beta) - \cos(\alpha)\sin(\beta) \\ &= -\frac{3}{8}\frac{3}{5} - \frac{-\sqrt{55}}{8}\frac{4}{5} \\ &= \frac{-9 + 4\sqrt{55}}{40} \end{aligned}$$

(f) $\cos(\alpha + \beta)$

Answer: First, we must fill out the β triangle. Because the adjacent side is 3 and hypotenuse is 5, the opposite will be 4, because that fills out the 3-4-5 right triangle. This is convenient. Then we have $\sin(\beta) = 4/5$. Here, we use

$$\cos(x + y) = \cos(x)\cos(y) - \sin(x)\sin(y)$$

for us, this means

$$\begin{aligned}\cos(\alpha + \beta) &= \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta) \\ &= \frac{-\sqrt{55}}{8} \frac{3}{5} - \frac{-3}{8} \frac{4}{5} \\ &= \frac{-3\sqrt{55} + 12}{40}\end{aligned}$$

14. Solve trigonometric equations on $[0, 2\pi)$

(a) $\sin(2x) + \sqrt{2}\cos(x) = 0$

Answer: We have that

$$\begin{aligned}2\sin(x)\cos(x) + \sqrt{2}\cos(x) &= 0 \\ \cos(x)(2\sin(x) + \sqrt{2}) &= 0 \\ \sin(x) &= -\frac{\sqrt{2}}{2}\end{aligned}$$

Which means we have solutions at $\frac{5\pi}{4}$ and $\frac{7\pi}{4}$. BUT, we also factored out $\cos(x)$, so we also have solutions when $\cos(x) = 0$, which is when $x = \pi/2, 3\pi/2$. Then, overall,

$$x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{4}, \frac{7\pi}{4}$$

(b) $2\sin^2(x) - 5\sin(x) + 2 = 0$

Answer: Let $u = \sin(x)$. Then we have

$$2u^2 - 5u + 2 = 0 \implies (2u - 1)(u - 2) \quad u = 2, u = 1/2$$

However, $\sin(x) = 2$ makes no sense given our possible ranges, so we solve x for $\sin(x) = 1/2$. This occurs at

$$x = \frac{\pi}{6}, \frac{5\pi}{6}$$

15. An aircraft is spotted by two observers who are 5000 meters apart. As the airplane passes over the line joining the observers, each observer takes a sighting of the angle of elevation of the airplane. The first observer sights the plane at 40° and the second observer sights the plane at 35° . How far away is the airplane from the first observer?

Answer: We have

$$\text{Law of Sines: } \frac{\sin(A)}{a} = \frac{\sin(B)}{b} = \frac{\sin(C)}{c} \quad (5.16)$$

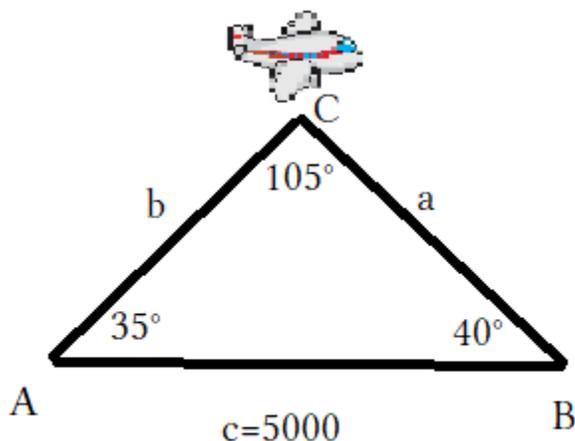


Figure 5.4: Diagram for aircraft question. Not to scale.

$$\begin{aligned}\sin(B)/b &= \sin(C)/c \\ b &= \frac{c \sin(B)}{\sin(C)} = 3327 \text{ meters}\end{aligned}$$

and

$$\begin{aligned}\sin(A)/a &= \sin(C)/c \\ a &= \frac{c \sin(A)}{\sin(C)} = 2969 \text{ meters}\end{aligned}$$

16. A tourist stands 100 feet from the base of the Leaning Tower of Pisa. With the tower leaning away from the observer, the observer looking up at an angle of 52° finds that the distance from the top of the tower to where he is standing is 228 feet. Find the angle the Leaning Tower makes with the ground.

Answer: We use the law of cosines, that

$$a^2 = b^2 + c^2 - 2bc \cos(A) \quad (5.17)$$

Plugging in yields (where we know we want a since it is opposite the one angle we do know)

$$a^2 = 100^2 + 228^2 - 2 \cdot 100 \cdot 228 \cos(52) \implies a = 184.13$$

We now use the law of sines:

$$\frac{\sin(52)}{184.14} = \frac{\sin(C)}{228} \implies \sin(C) = 0.976$$

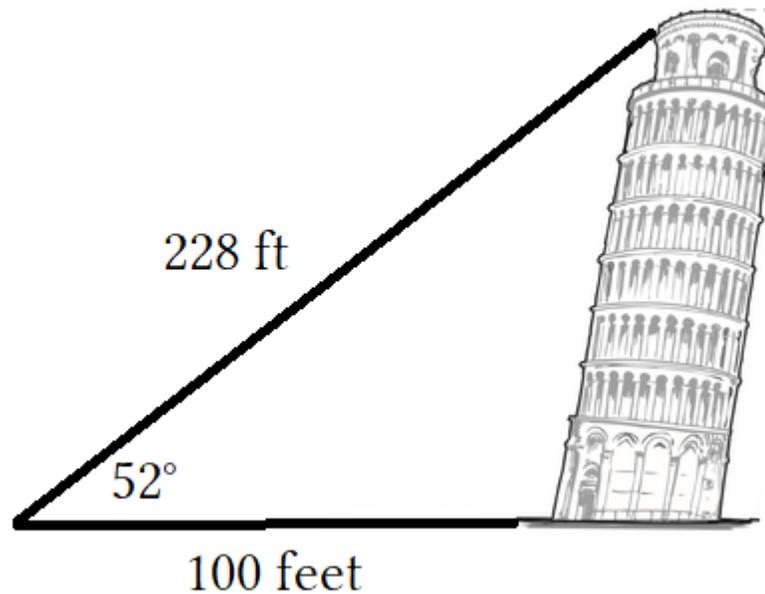


Figure 5.5: Diagram for leaning tower of Pisa question. Not to scale.

We have the answers $C = 77.34^\circ$ and $C = 102.7^\circ$, i.e. answers in quadrant 1 and quadrant 2. We keep the second answer here because we know the tower is leaning away, and thus is greater than 90 degrees¹

¹We are also limited by the fact that a triangle angle must be between 0 and 180 degrees, among other restrictions.

17. Find the power series representation $\frac{4x^3}{(x-6)^2}$

Answer: Notice that $(x-6)^2 = (6-x)^2$, then we have

$$\frac{4x^3}{(x-6)^2} = \frac{4x^3}{(6-x)^2}$$

Which we further simplify to

$$\frac{4x^3}{(x-6)^2} = \frac{4x^3}{6^2(1-\frac{x}{6})^2} = \frac{x^3}{9(1-\frac{x}{6})^2}$$

We can expand this with a power series:

$$\frac{x^3}{9(1-\frac{x}{6})^2} = \frac{1}{9} \sum_{n=1}^{\infty} nx^{n-1} x^3 \left(\frac{1}{6}\right)^{n-1} = \boxed{\frac{1}{9} \sum_{n=1}^{\infty} nx^{n+2} \left(\frac{1}{6}\right)^{n-1}}$$

where we made $x/6 = x \cdot (1/6)$ because only the x -exponent adds with the x^3 .