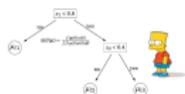


# Chapter 10 Notes



Conditional Probability, Independent  
Events, Bayes' Formula  
STP-231

Arizona State University



## Independence

- **Dependent events:** The occurrence of one event **changes** the probability of a different event occurring
- Example: The probability Riley gets an A on their test depends on whether or not they study more than 3 hours
- **Independent events:** The occurrence of event does not change the probability of a second event occurring
- Example: We assume the probability of a flipping a coin tails is independent of the previous flip



## Independence Continued

- If events  $A$  and  $B$  are independent, then:

$$\Pr(A \text{ and } B) = \Pr(A) \times \Pr(B)$$

- If they are not independent, then:

$$\Pr(A \text{ and } B) \neq \Pr(A) \times \Pr(B)$$

- Independence is often assumed in a probability model where the events seem to have no connection. Also, its convenient and can make the problem easier, though we need just justification to claim independence!



## Mutually Exclusive vs Independent Events Not the same!

- Example:  $\Rightarrow$  Roll a die,  $A$  is event of rolling a 2,4, or 6, and event  $B$  is rolling a 6. Since the rolls are independent, this is an independent event, i.e.

$$\Pr(A \text{ and } B) = \Pr(A) \Pr(B) = \frac{1}{2} \times \frac{1}{6} = \frac{1}{12} \neq 0$$

Independent, but not mutually exclusive

- Example:  $\Leftarrow$  Again roll a die. Let  $A$  be event your roll is an odd, and event  $B$  you roll a 6. These are mutually exclusive, but

$$\Pr(B) = 1/6 \quad \Pr(A) = 1/2 \text{ but } \Pr(A \cap B) = 0$$

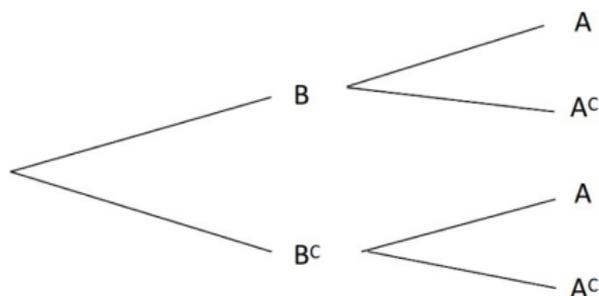
But these events are dependent, as rolling a 6 means you roll an even, and event  $A$  means you rolled an odd, so these two can't happen simultaneously. In general, mutually exclusive events are dependent



## Conditional Probability

- Probability of an event occurring given that another event has already occurred:

$$\Pr(A | B) = \frac{\Pr(A \text{ and } B)}{\Pr(B)}, \quad \text{provided } \Pr(B) > 0$$



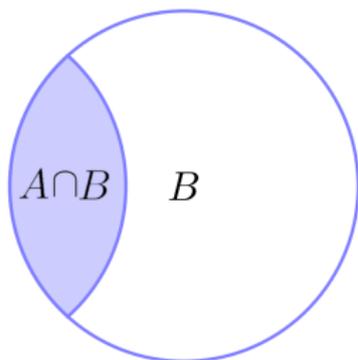
- Example: Roll fair dice once. Given an odd number was rolled, what is the probability of rolling a one. Given an even number was rolled, what is the probability of rolling a two or six?



## Venn Diagram Approach

We know with certainty the  $B$  event did occur, so the probability of  $A$  in this situation is the shaded region in the picture (the only place where  $A$  occurs given that  $B$  for sure occurred) divided by the total probability of  $B$ , i.e.

$$\Pr(A | B) = \frac{\Pr(A \text{ and } B)}{\Pr(B)}$$



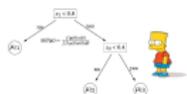


## Example

The culinary herb cilantro is very polarizing: Some love it and others hate it (some say it tastes like soap). Here are the results from a survey asking 12087 people whether or not they like/dislike cilantro

	# of Men	# of Women
Dislikes cilantro	1632	1549
Likes cilantro	5165	3741

- What is the approximate conditional probability that an adult likes cilantro given they are a man
- What is the approximate probability this person is a man and likes cilantro?



## Another Example

- Your friend tells you they will visit you one week-day in the coming week, with a probability of 0.20 of visiting on any given week day.
- Given that it is Thursday and your friend has yet to visit, what is the probability they will visit today?

$$\begin{aligned}
 \Pr(A \mid B) &= \frac{\Pr(A \cap B)}{\Pr(B)} \\
 &= \frac{\Pr(\text{visit today and not M, T, or W})}{\Pr(\text{did not visit M, T, or W})} \\
 &= \frac{0.20}{1 - 0.60} = 0.50
 \end{aligned}$$



## Conditional Probability Rules

- General multiplication rule: the probability of events  $A$  and  $B$  happening together is (both are equivalent):

$$\Pr(A \text{ and } B) = \Pr(A) \times \Pr(B | A) \quad \text{or} \quad \Pr(B) \times \Pr(A | B)$$

- We also know from earlier rules that:

$$\Pr(A^c | B) = 1 - \Pr(A | B)$$

$$\Pr(A | B) = 1 - \Pr(A^c | B)$$

- If two events  $A$  and  $B$  are independent, then

$$\Pr(A | B) = \Pr(A)$$



## Diagnostic Tests

- True Positive: The test states that the person/item has the disease when they really have the disease
- False Negative: The test states that the person/item does not have the disease when they do have the disease
- False Positive: The test states that the person/item has the disease when they really do not have the disease
- True Negative: The test states that the person/item does not have the disease when they do not have the disease



## Diagnostic Tests Example

- Suppose a medical test has a 92% chance of detecting a disease given the person has the disease and a 94% chance of correctly indicating that the disease is absent if the person really does not have the disease. Suppose 10% of the population has the disease
- What is the probability that a randomly chosen person will test positive?

$$0.10 * 0.92 + (1 - 0.94) * (1 - 0.10) = 0.146 = 14.6\%$$

- Suppose a randomly chosen person does test positive. What is the probability that this person really has the disease?

$$0.10/0.146 = 0.685 = 68.5\%$$



## Contingency Tables

- Useful to display bivariate data. Also helpful to determine prediction accuracy, to calculate joint and conditional probabilities, and see balance in data outcomes<sup>1</sup>

Sleep < 7 hours	Have young kid		Total sleep
	Yes	No	
Yes	21	122	143
No	70	493	563
Total Had kid	91	615	706

<sup>1</sup>Data from Wooldridge Economics



## Contingency Tables Example

In a study of the relationship between health risk and income, a large group of people living in Massachusetts were asked a series of questions. The following data table is taken from the study, relating to the comparison between income and the amount of stress reported by the people in the study

	Income			Total
	Low	Medium	High	
Stressed	526	274	216	1,016
Not Stressed	1,954	1,680	1,899	5,533
Total	2,480	1,954	2,115	6,549

- $\Pr(\text{Low Income})$
- $\Pr(\text{Stressed and Low Income})$
- $\Pr(\text{Stressed} \mid \text{Low Income})$



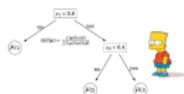
## Contingency Tables

### Another Example

Random sample of residents was selected and each response was categorized according to revenue preference and age. The questions were related to the legalization of gambling.

Age	Gambling	Liquor Stores	Other	Total
18-20	33	68	12	113
21-30	55	121	50	226
31-44	117	109	132	358
At least 45	158	110	90	358
Total	363	408	284	1055

- What is probability resident is in favor of legalized gambling?
- What is probability person is 31-44 given they are in favor of state-owned liquor stores?
- What is the probability person is 21-30 and in favor of other?
- What is the probability of being 18-20 or 45+?
- Are the events favoring liquor stores and being 31-44

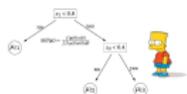


## Example

Smoking Status	Upper	Middle	Lower	Total
Current	51	22	43	116
Former	55	121	50	141
Never	117	109	132	99
Total	211	52	93	356

- $\Pr(\text{Current smoker} \mid \text{upper class}) = \frac{51}{211}$
- $\Pr(\text{never smoked} \ \& \ \text{lower class}) = \frac{22}{356}$
- $\Pr(\text{middle class} \mid \text{former smoker}) = \frac{21}{141}$
- $\Pr(\text{former smoker}) = \frac{141}{356}$
- $\Pr(\text{Upper class} \ \& \ \text{current smoker}) = \frac{51}{356}$
- $\Pr(\text{Upper class}) \times \Pr(\text{Current smoker}) = \frac{211}{356} \times \frac{116}{356} \approx 0.19$

# Bayes Theorem



## Law of Total Probability For two Events

For two events  $A$  and  $B$

$$\Pr(A) = \Pr(B) \times \Pr(A | B) + \Pr(B^c) \times \Pr(A | B^c)$$

$$\Pr(B) = \Pr(A) \times \Pr(B | A) + \Pr(A^c) \times \Pr(B | A^c)$$



## Bayes' Theorem

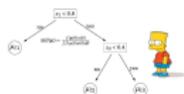
Suppose that  $A_1, A_2, \dots, A_k$  are disjoint events whose probabilities are not 0 and sum to 1

That is, any outcome has to be exactly in one of these events.

Then if  $B$  is any other event whose probability is not 0 or 1, then

$$\Pr(A_i | B) = \frac{\Pr(B|A_i) \Pr(A_i)}{\Pr(B|A_1) \Pr(A_1) + \Pr(B|A_2) \Pr(A_2) + \dots + \Pr(B|A_k) \Pr(A_k)}$$

Notice, the  $i$  index in the numerator is not the same as the index in the denominator!



## Bayes' Theorem

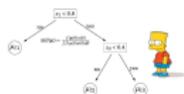
This comes from conditional probability:

$$\Pr(A | B) = \frac{\Pr(A \text{ and } B)}{\Pr(B)} \quad (1)$$

$$\Pr(B | A) = \frac{\Pr(A \text{ and } B)}{\Pr(A)} \implies \Pr(A \text{ and } B) = \Pr(B | A) \Pr(A) \quad (2)$$

$$\implies \Pr(A | B) = \frac{\Pr(B | A) \Pr(A)}{\Pr(B)} \quad (2) \text{ in num, } (1) \text{ in denom} \quad (3)$$

But,  $\Pr(B) = \sum_{j=1}^k \Pr(A_j \cap B) = \sum_{j=1}^k \Pr(B | A_j) \Pr(A_j)$   
 (from law of total probability) which is how we get the theorem  
 (notice we index over  $j$  not  $i$ , because we are summing over all  
 the possible events  $A$  we can condition on)



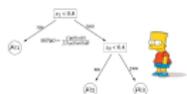
## Bayes' Theorem

If we are given  $\Pr(B)$ , then:

$$\Pr(A | B) = \frac{\Pr(B | A) \Pr(A)}{\Pr(B)}$$

However, even if we don't know  $\Pr(B)$ , we're okay. If we let  $A_i$  we just have two options, an event and its compliment, i.e.  $A$  and  $A^c$ ,

$$\Pr(A | B) = \frac{\Pr(B | A) \Pr(A)}{\Pr(B | A) \Pr(A) + \Pr(B | A^c) \Pr(A^c)}$$



## An Example

Suppose:

$$\Pr(\text{dangerous fire}) = 0.01$$

$$\Pr(\text{smoke}) = 0.10$$

$$\Pr(\text{A dangerous fire makes smoke}) = 0.90$$

What is the probability of dangerous fire when there is smoke?

$$\begin{aligned} \Pr(\text{Fire} \mid \text{Smoke}) &= \frac{\Pr(\text{Fire}) \Pr(\text{Smoke} \mid \text{Fire})}{\Pr(\text{Smoke})} \\ &= \frac{0.01 \cdot 0.90}{0.10} \\ &= 0.09 \end{aligned}$$



## Examples

Suppose that a medical test has a 99% chance of detecting a disease given the person actually has the disease. The test has a 90% chance of correctly telling someone they do not have the disease when they in fact do not have the disease, i.e. 10% of people are falsely told they have the disease. Now, suppose 5% of the population actually has the disease.

- What is the probability that a randomly chosen person will test positive?

$$\begin{aligned}
 \Pr(P) &= \Pr(P \mid D) \Pr(D) + \Pr(P \mid D^c) \Pr(D^c) \\
 &= 0.99 \cdot 0.05 + 0.10 \cdot 0.95 \\
 &= 0.1445
 \end{aligned}$$

i.e. even though only 5% of people have disease, 14.45% of tests come back positive!



## Example Continued

- Suppose that a person does test positive. What is the probability that this person really has the disease?
- We still need to account for false negatives in the numerator, but basically we are dividing the number of people who really have the disease over the total number of positive

$$\Pr(D | P) = \frac{\Pr(P | D) \Pr(D)}{\Pr(P | D) \Pr(D) + \Pr(P | D^c) \Pr(D^c)}$$

- So most of error comes from rarity of disease and false positives not the test missing the disease



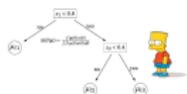
## Extra Example

Overall, suppose  $1/4$  of students get a B on an exam. Now suppose  $2/3$  of students do not carefully read exam questions, and in that case  $1/5$  of them get a B.

- Prob a student who read instructions correctly gets a B:  
Let  $B$  be the event of getting a B, and  $R$  be the event of reading instructions.

$$\Pr(B | R) = \frac{\frac{1}{4} - \frac{1}{5} \cdot \frac{2}{3}}{\frac{1}{3}} = \frac{7}{20} = 0.35$$

- The numerator is the proportion of total students who got a B minus the proportion who got a B while not reading the instructions. This gives us the proportion of students who got a B while reading correctly. However, we divide by the prob a student read correctly to get the probability given that conditional



## $\Pr(B|R)$ mathematically

- We want  $\Pr(B | R)$ . Bayes theorem applied directly gives us

$$\Pr(B | R) = \frac{\Pr(B) \Pr(R | B)}{\Pr(R)}$$

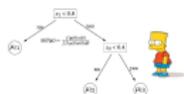
But we don't know  $\Pr(R | B)$ .

- Recall, (see slides 16-18)

$$\Pr(B) = \Pr(B | R) \Pr(R) + \Pr(B | R^c) \Pr(R^c)$$

so that means, solving for  $\Pr(B | R)$ :

$$\Pr(B | R) = \frac{\Pr(B) - \Pr(B | R^c) \Pr(R^c)}{\Pr(R)}$$



## Continued

- Prob a student who gets a B has read the instructions correctly:

$\Pr(B) = 1/4$ ,  $\Pr(B | R^c) = (1/5) \cdot (2/3) = 2/15$ . Using Bayes rule:

$$\Pr(R | B) = \frac{\Pr(B|R) \Pr(R)}{\Pr(B|R) \Pr(R) + \Pr(B|R^c) \Pr(R^c)} = \frac{(7/20) \cdot (1/3)}{(7/20) \cdot (1/3) + (1/5) \cdot (2/3)} = \frac{7}{15}$$